

# ADAMS COMPLETION FOR COHOMOLOGY THEORIES ARISING FROM KAN EXTENSIONS

SRIBATSA NANDA

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## Abstract

It is shown that a cohomology theory over an admissible category, which is obtained from an additive cohomology theory over a smaller admissible category, through the Kan extension process, always admits global Adams completion.

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## 1. Introduction

Let  $\mathcal{C}$  be an arbitrary category and  $S$  a set of morphisms of  $\mathcal{C}$ . Let  $\mathcal{C}[S^{-1}]$  denote the category of fractions of  $\mathcal{C}$  with respect to  $S$ ; then for a given object  $Y$  of  $\mathcal{C}$ ,

$$\mathcal{C}[S^{-1}](-, Y) : \mathcal{C} \rightarrow \text{Sets}$$

defines a contravariant functor. If this functor is representable by an object  $Y_S$  of  $\mathcal{C}$ , that is,

$$\mathcal{C}[S^{-1}](-, Y) \cong \mathcal{C}(-, Y_S)$$

then  $Y_S$  is called the (generalized) Adams completion of  $Y$  with respect to the set of morphisms  $S$  or simply the  $S$ -completion of  $Y$  (see Deleanu and others (1972), Adams (1973)). We shall often refer to  $Y_S$  simply as the completion of  $Y$ .

In particular, let  $\mathcal{C}$  be a full subcategory of the category of based topological spaces and based continuous maps such that (a) it contains singletons (b) it contains entire based homotopy types and which moreover has the following property. Let  $f : X \rightarrow Y$  be in  $\mathcal{C}$ ; let  $i_f : X \rightarrow M_f$  denote the canonical inclusion of  $X$  into  $M_f$ , the (reduced) mapping cylinder of the map  $f$ . Clearly,  $i_f$  is in  $\mathcal{C}$ . Given a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \\ Z & & \end{array}$$

in  $\mathcal{C}$ , we form the push-out of  $g$  and  $i_f$  in the category of all spaces :

$$\begin{array}{ccc} X & \xrightarrow{i_f} & M_f \\ g \downarrow & & \downarrow k \\ Z & \xrightarrow{u} & W \end{array}$$

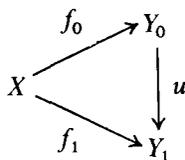
We assume that (c)  $\mathcal{C}$  contains the push-out diagram above. Such a category will be called admissible. It is clear that an admissible category is closed under the formation of mapping cones.

Let  $\mathcal{C}$  be an admissible category as above and  $\tilde{\mathcal{C}}$  its homotopy category. Let  $h$  be a generalized homology (cohomology) theory defined on  $\tilde{\mathcal{C}}$ . Let  $S$  be the set of morphisms of  $\tilde{\mathcal{C}}$  which are carried into isomorphisms by  $h$ . If every object of  $\tilde{\mathcal{C}}$  admits a completion with respect to  $S$ , then we say that the homology (cohomology) theory  $h$  admits global Adams completion. Deleanu (1974) has shown that any additive theory  $h$  on the homotopy category of based CW-complexes and based continuous maps admits global Adams completion. In this note, we show that every cohomology theory on an admissible category arising from an additive cohomology theory on a smaller admissible category through the Kan extension process (see Deleanu and Hilton (1968, 1971), Hilton (1968)) always admits global Adams completion. More precisely, let  $\mathcal{C}_0$  and  $\mathcal{C}_1$  be admissible cocomplete categories with  $\mathcal{C}_0 \subset \mathcal{C}_1$ . Let  $h_0$  be an additive cohomology theory on  $\tilde{\mathcal{C}}_0$  such that its Kan extension  $h_1$  over  $\tilde{\mathcal{C}}_1$  is also a cohomology theory; then  $h_1$  admits global Adams completion. Deleanu and Hilton (1968, 1971) have given a number of examples of cohomology theories arising through the Kan extension process. An important consequence of this result is that all those theories admit global Adams completion. The proof of this result depends mainly on the particularly nice description of the cohomology group  $h_1^n(X)$  as given by Hilton (1968) and additivity of the functor  $h_0$ . In the next section, we briefly review this description and in Section 3, we present the proof of the main result.

## 2. Description of the groups $h_1^n(X)$

Let  $h_0$  be a given additive cohomology theory on  $\tilde{\mathcal{C}}_0$ . For a given object  $X$  of  $\tilde{\mathcal{C}}_1$ , consider the category of  $\tilde{\mathcal{C}}_0$ -objects under  $X$ : an object in this category is a

morphism  $f: X \rightarrow Y$  in  $\tilde{\mathcal{C}}_1$  where  $Y$  is an object of  $\tilde{\mathcal{C}}_0$ . A morphism  $u$  in this category  $u: f_0 \rightarrow f_1$  is a commutative diagram

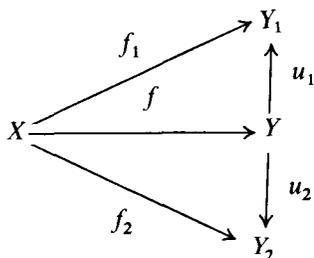


The Kan extension  $h_1$  of the cohomology theory  $h_0$  is now defined as follows :

$$h_1^n(X) = \lim_{\overleftarrow{f}} (h_0^n(Y), h_0^n(u)).$$

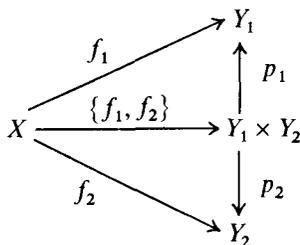
Obviously,  $h_1^n$  is an extension of  $h_0^n$  and defines a contravariant functor on  $\tilde{\mathcal{C}}_1$  with values in the category of abelian groups. We suppose that this extension defines a cohomology theory on  $\tilde{\mathcal{C}}_1$ . For various conditions under which this happens, see Deleanu and Hilton (1968, 1971), Hilton (1968).

We now drop the superscript  $n$  for convenience and give an alternative description of the groups  $h_1(X)$  (see, for example, Hilton (1968)). Consider the set of all pairs  $(\alpha, f)$  where  $f: X \rightarrow Y$  is in  $\tilde{\mathcal{C}}_1$ ,  $Y$  an object of  $\tilde{\mathcal{C}}_0$  and  $\alpha \in h_0(Y)$ . Introduce a relation  $(\alpha_1, f_1) \sim (\alpha_2, f_2)$  if there is a diagram

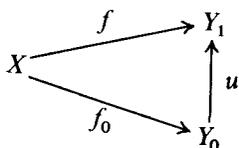


with  $u_1, u_2 \in \tilde{\mathcal{C}}_0$ ,  $u_1 f = f_1$ ,  $u_2 f = f_2$  and  $u_1^*(\alpha_1) = u_2^*(\alpha_2)$  where  $u_i^* = h_0(u_i)$ . This is an equivalence relation; we denote by  $[\alpha, f]$  the equivalence class of  $(\alpha, f)$ . Now define addition of these classes by the rule

$$[\alpha_1, f_1] + [\alpha_2, f_2] = [p_1^*(\alpha_1) + p_2^*(\alpha_2), \{f_1, f_2\}]$$



where  $p_1, p_2$  are the respective projections. That this rule of addition is independent of the representatives chosen, can be easily checked. With respect to this addition, the set  $E(X)$  of all equivalence classes forms an abelian group. We note, in particular, that a class  $[\alpha, f] = 0$  if and only if there is a commutative diagram



in  $\tilde{\mathcal{C}}_1$  with  $u \in \tilde{\mathcal{C}}_0$  such that  $u^*(\alpha) = 0$ . Now define  $i_f : h_0(Y) \rightarrow E(X)$  by the rule

$$i_f(\alpha) = [\alpha, f].$$

The group  $E(X)$  together with the homomorphisms  $\{i_f\}$  enjoys the universal mapping property by which a direct limit is defined. We therefore identify  $E(X)$  with

$$\varinjlim (h_0(Y), h_0(u)) = h_1(X).$$

Note also that if  $g : X \rightarrow X'$  is a morphism in  $\tilde{\mathcal{C}}_1$ , then  $h_1(g) = g^* : h_1(X') \rightarrow h_1(X)$  is defined by the rule :

$$g^*[\alpha, f] = [\alpha, fg]$$

### 3. Adams completion for the cohomology theory $h_1$

We assume throughout that both  $\tilde{\mathcal{C}}_0$  and  $\tilde{\mathcal{C}}_1$  are admissible and cocomplete and  $\tilde{\mathcal{C}}_1$  is a small  $\mathcal{U}$ -category, where  $\mathcal{U}$  is a fixed Grothendieck universe, that is (i) the set of objects of  $\tilde{\mathcal{C}}_1$  form a subset of  $\mathcal{U}$  and (ii) for any pair of objects  $X$  and  $Y$  of  $\tilde{\mathcal{C}}_1$ , the set  $\text{Hom}_{\tilde{\mathcal{C}}_1}(X, Y)$  of all morphisms of  $\tilde{\mathcal{C}}_1$  from  $X$  to  $Y$  is a set in  $\mathcal{U}$ . Recall that Deleanu (1975) has given two conditions for a cocomplete small  $\mathcal{V}$ -category  $\mathcal{C}$  to admit global Adams completion with respect to a set of morphisms  $T$  that admits a calculus of left fractions. One of these conditions (condition  $(*)$  of Deleanu's main theorem (Deleanu (1975)) is imposed only to ensure that the category of fractions  $\mathcal{C}[T^{-1}]$  remains within the given initial universe. We have shown that this condition is automatically satisfied (see Nanda (1979)). Thus we have to check only one condition which shall be referred to as the *compatibility axiom* : If  $\{t_\alpha : X_\alpha \rightarrow X'_\alpha\}_{\alpha \in L}$  is a set of morphisms of  $\mathcal{C}$  such that each  $t_\alpha \in T$  and  $L \in \mathcal{V}$ , then

$$\bigvee_{\alpha \in L} t_\alpha : \bigvee_{\alpha \in L} X_\alpha \rightarrow \bigvee_{\alpha \in L} X'_\alpha \text{ is in } T.$$

We now apply this result to the category  $\tilde{\mathcal{C}}_1$ . Let  $S$  be the set of morphisms of  $\tilde{\mathcal{C}}_1$  which are carried into isomorphisms by the cohomology functor  $h_1$ . It follows easily

from Theorem 3.1 of Deleanu and others (1972) (by replacing homology by cohomology, throughout the proof) that  $S$  admits a calculus of left fractions. We start by proving the following

**PROPOSITION 3.1.** *The cohomology theory  $h_1$  satisfies the wedge axiom.*

**PROOF.** We denote by  $i_\alpha$  the homotopy class of the inclusion map  $X_\alpha \rightarrow \bigvee_{\beta \in L} X_\beta$ ,  $\alpha \in L$ . The map  $i_\alpha$  induces homomorphisms

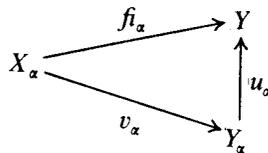
$$i_\alpha^* = h_1(i_\alpha) : h_1(\bigvee_{\beta \in L} X_\beta) \rightarrow h_1(X_\alpha).$$

We have to show that the homomorphism

$$\{i_\alpha^*\} : h_1(\bigvee_{\beta \in L} X_\beta) \rightarrow \prod_\alpha h_1(X_\alpha)$$

is an isomorphism.

Suppose that  $[w, f]$  is a class in  $h_1(\bigvee X_\alpha)$  such that  $\{i_\alpha^*\}([w, f]) = 0$ . We then have a map  $f : \bigvee X_\alpha \rightarrow Y$  in  $\tilde{\mathcal{C}}_1$  with  $Y \in \tilde{\mathcal{C}}_0$  and  $w \in h_0(Y)$  such that  $[w, f_\alpha] = 0$  for every  $\alpha \in L$ . This implies that for each  $\alpha \in L$ , there is a space  $Y_\alpha$  in  $\tilde{\mathcal{C}}_0$  and maps  $u_\alpha : Y_\alpha \rightarrow Y$  in  $\tilde{\mathcal{C}}_0$  and  $v_\alpha : X_\alpha \rightarrow Y_\alpha$  in  $\tilde{\mathcal{C}}_1$  such that  $u_\alpha^*(w) = h_0(u_\alpha)(w) = 0$  and the following diagram is commutative :



Let  $j_\alpha : Y_\alpha \rightarrow \bigvee_{\beta \in L} Y_\beta$  denote the homotopy class of the inclusion map. We can then factorise  $u_\alpha = t j_\alpha$  with  $t : \bigvee_{\alpha \in L} Y_\alpha \rightarrow Y$ . Thus, for each  $\alpha \in L$ ,

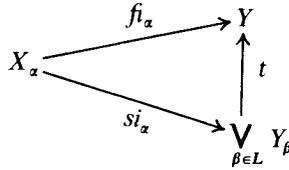
$$0 = u_\alpha^*(w) = j_\alpha^* t^*(w)$$

so that

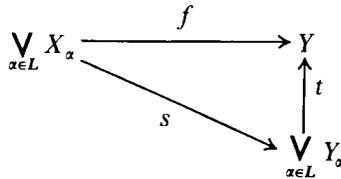
$$\{j_\alpha^*\} : h_0(\bigvee_{\alpha \in L} Y_\alpha) \rightarrow \prod_{\alpha \in L} h_0(Y_\alpha)$$

maps the element  $t^*(w)$  to zero. By assumption, the theory  $h_0$  is additive, so that  $\{j_\alpha^*\}$  is an isomorphism; thus  $t^*(w) = 0$ .

Now consider the maps  $j_\alpha v_\alpha : X_\alpha \rightarrow \bigvee Y_\alpha$ . By the universal property of the wedge, each  $j_\alpha v_\alpha$  can be factorized as  $j_\alpha v_\alpha = s i_\alpha$  where  $s : \bigvee X_\alpha \rightarrow \bigvee Y_\alpha$ . Consider the diagram

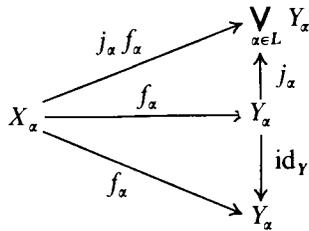


We have  $tsi_\alpha = tj_\alpha v_\alpha = u_\alpha v_\alpha = fi_\alpha$ ; thus the diagram is commutative. It follows from the universal property of the wedge that  $ts = f$ . Thus we have a commutative diagram

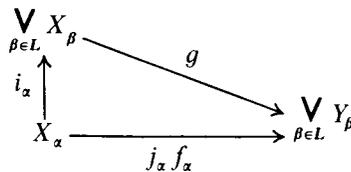


with  $t \in \tilde{\mathcal{C}}_0$  and  $t^*(w) = 0$ ; thus  $[w, f] = 0$  implying that  $\{i_\alpha^*\}$  is injective.

To show that  $\{i_\alpha^*\}$  is surjective, let  $\{[w_\alpha, f_\alpha]\}_{\alpha \in L}$  be an element of  $\prod_{\alpha \in L} h_1(X_\alpha)$ . For each  $\alpha \in L$ , let the class  $[w_\alpha, f_\alpha]$  be represented by  $f_\alpha : X_\alpha \rightarrow Y_\alpha$  with  $Y_\alpha \in \tilde{\mathcal{C}}_0$  and  $w_\alpha \in h_0(Y_\alpha)$ . The element  $\{w_\alpha\} \in \prod_{\alpha \in L} h_0(Y_\alpha)$  corresponds to some element  $w \in h_0(\bigvee_{\alpha \in L} Y_\alpha)$  under the isomorphism  $\{j_\alpha^*\}$  since  $h_0$  satisfies the wedge axiom. Thus  $j_\alpha^*(w) = w_\alpha$ . Consider now the diagram



Clearly, the two triangles are commutative. Thus for each  $\alpha \in L, [w, j_\alpha f_\alpha] = [w_\alpha, f_\alpha]$ . The map  $j_\alpha f_\alpha : X_\alpha \rightarrow \bigvee_{\beta \in L} Y_\beta$  factors through  $\bigvee_{\alpha \in L} X_\alpha$  giving rise to a commutative diagram



Consider the class  $[w, g] \in h_1(\bigvee_{\alpha \in L} X_\alpha)$ . We have

$$i_\alpha^*[w, g] = [w, gi_\alpha] = [w, j_\alpha f_\alpha] = [w_\alpha, f_\alpha]$$

for each  $\alpha \in L$ . Thus

$$\{i_\alpha^*\} [w, g] = \{[w_\alpha, f_\alpha]\}_{\alpha \in L}$$

showing that  $\{i_\alpha^*\}$  is an epimorphism.

**THEOREM 3.2.** *The cohomology theory  $h_1$  admits global Adams completion.*

**PROOF.** It suffices to prove that  $h_1$  satisfies the compatibility axiom. Let

$$\{s_\alpha : X_\alpha \rightarrow X'_\alpha, \alpha \in L\} \subset S$$

such that the index  $L \in \mathcal{U}$ . We have to show that

$$\mathbf{V} s_\alpha : \mathbf{V}_{\alpha \in L} X_\alpha \rightarrow \mathbf{V}_{\alpha \in L} X'_\alpha$$

is in  $S$ . Consider the diagram

$$\begin{array}{ccc}
 h_1(\mathbf{V}_{\alpha \in L} X_\alpha) & \xrightarrow{\{i_\alpha^*\}} & \prod_{\alpha \in L} h_1(X_\alpha) \\
 (\mathbf{V} s_\alpha)^* \uparrow & & \uparrow \prod_{\alpha \in L} s_\alpha^* \\
 h_1(\mathbf{V}_{\alpha \in L} X'_\alpha) & \xrightarrow{\{i'_\alpha^*\}} & \prod_{\alpha \in L} h_1(X'_\alpha)
 \end{array}$$

Since each  $s_\alpha^*$  is an isomorphism, so is  $\prod_{\alpha \in L} s_\alpha^*$ ; Proposition 3.1 implies that the horizontal arrows are isomorphisms too. It follows, therefore, that  $(\mathbf{V} s_\alpha)^*$  is an isomorphism and the proof is complete.

### 4. Some final remarks

We conclude this note with some general remarks on the existence of the Adams completion. Suppose that  $\mathcal{C}$  is a cocomplete admissible small  $\mathcal{U}$ -category and  $h$  a cohomology theory defined on  $\mathcal{C}$ . Let (A) and (B) and (C) describe the following conditions.

- (A)  $h$  satisfies the wedge axiom.
- (B)  $h$  satisfies the compatibility axiom.
- (C)  $h$  admits global Adams completion.

It is clear from the diagram of Theorem 3.2 that (A)  $\Rightarrow$  (B). Deleanu's result (1975, Theorem 1) implies that (B)  $\Rightarrow$  (C). It is also proved in Theorem 2 that (C)  $\Rightarrow$  (B). We thus have

$$(A) \Rightarrow (B) \Leftrightarrow (C).$$

We do not know whether (B)  $\Rightarrow$  (A). It is our conjecture that this is true. Were it true, it would imply that only additive cohomology theories would admit global Adams completion; thus, it would settle the question ‘which cohomology theories on an admissible category admit global Adams completion’.

Similar considerations also apply if  $h$  is a homology theory on  $\tilde{\mathcal{C}}$ . Suppose that  $h$  satisfies the wedge axiom : For every collection  $\{X_\alpha\}_{\alpha \in L}$  of spaces in  $\mathcal{C}$ , the inclusions  $i_\alpha : X_\alpha \rightarrow \bigvee_{\beta \in L} X_\beta$  induce an isomorphism

$$\{i_{\alpha^*}\} : \bigoplus_{\alpha \in L} h(X_\alpha) \rightarrow h(\bigvee_{\alpha \in L} X_\alpha).$$

Let  $s_\alpha : X_\alpha \rightarrow X'_{\alpha^*}$ ,  $\alpha \in L$  be a family of morphisms of  $\tilde{\mathcal{C}}$  such that each  $s_{\alpha^*}$  is an isomorphism. The commutative diagram

$$\begin{array}{ccc} \bigoplus_{\alpha} h(X_\alpha) & \xrightarrow{\{i_{\alpha^*}\}} & h(\bigvee_{\alpha \in L} X_\alpha) \\ \downarrow \bigoplus_{\alpha} s_{\alpha^*} & & \downarrow (\bigvee_{\alpha \in L} s_{\alpha^*})^* \\ \bigoplus_{\alpha} h(X'_{\alpha^*}) & \xrightarrow{\{i'_{\alpha^*}\}} & h(\bigvee_{\alpha \in L} X'_{\alpha^*}) \end{array}$$

shows that  $(\bigvee_{\alpha \in L} s_{\alpha^*})^*$  is an isomorphism, so that the wedge axiom implies the compatibility axiom. Thus, in case of homology we also have

$$(A) \Rightarrow (B) \Leftrightarrow (C).$$

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Mathematics Department  
 Regional Engineering College  
 Rourkela 769008 (Orissa), India