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Miscellaneous Topics Related to Rectifiability

Here I only briefly present some other topics related to rectifiability.

16.1 Curvature Measures

Federer introduced in [202] sets with positive reach and curvature measures. This paper has had and is still having a huge impact on matters related to convexity and integral geometry. Sets with positive reach include both convex sets and C^2 submanifolds. A closed set $F \subset \mathbb{R}^n$ has positive reach if there is $r > 0$ such that if $d(x, F) \leq r$, then there is a unique $\pi_F(x) \in F$ with $|\pi_F(x) - x| = d(x, F)$.

Federer proved that there are signed Borel measures $\mu_i(F, \cdot)$, called *curvature measures*, such that for every Borel set $B \subset \mathbb{R}^n$ and for $r > 0$ as above

$$\mathcal{L}^n(\{x\colon d(x, F) \leq r, \pi_F(x) \in B\}) = \sum_{i=0}^{n} \alpha(n-i) r^{n-i} \mu_i(F, B).$$

For convex sets F this is Minkowski's well-known Quermassintegrale formula.

In addition, Federer proved generalizations of the Gauss–Bonnet formula of differential geometry and the principal kinematic formula of integral geometry. His methods were partially based on his 1947 rectifiability theory. This paper provided one of the first applications of this theory.

In [445], Zähle gave a representation of curvature measures in terms of rectifiable currents supported by the unit normal bundle of F, which is an $(n-1)$-rectifiable subset of $\mathbb{R}^{2n}$. Rataj and Zähle continued this work in several papers, see, for example, [387] and their monograph [388], which also gives a wider presentation of the topic.

16.2 Dynamical Systems

For many dynamical systems the following dichotomy is typical: either the limit set is a fractal or something very special, for example, a piece of a plane, a sphere or a real or complex analytic set. Such limit sets include self-similar and self-conformal sets, Julia sets of rational functions and limit sets of Kleinian groups. Fractal here could mean how Mandelbrot at one point defined fractal: Hausdorff dimension is strictly bigger than topological dimension. Mayer and Urbański [333] and Das, Simmons and Urbański [131] used rectifiability to prove this type of results. In the paper [131], the setting is very general including the above cases as special cases, even in infinite-dimensional Hilbert spaces.

We describe the procedure vaguely (and with errors). Let K be such a limit set. In all cases there is some self-similarity present: there are appropriate maps (similarities, conformal, and so on) that map small subsets of K onto its large subsets. One should show that if the Hausdorff dimension $m = \dim K$ equals the topological dimension $\dim_T K$, then K is something very special. First, the system offers some type of invariant measure, which can be related to $\mathcal{H}^m$ by scaling properties and then one has $\mathcal{H}^m(K) < \infty$. By [200], the assumption $\dim_T K = m$ implies that many projections of K on m-planes have positive $\mathcal{H}^m$ measure. Hence by the Besicovitch–Federer projection Theorem 4.17, K is not purely m-unrectifiable, so it has approximate tangent planes at many points x by Theorem 4.5. The appropriate maps send the planes to the very special sets we are after. Using these maps to blow up small neighbourhoods of x to large sets and taking a limit concludes the proof.

This approach does not work in infinite-dimensional spaces. In particular, the projection theorem is false (recall Section 7.6). The authors of [131] deal with this, extending the family of rectifiable sets to what they call pseudorectifiable. A set E with $\mathcal{H}^m(E) < \infty$ is pseudorectifiable if there are m-planes $T_E(x)$ and a measure $\mu_E \sim \mathcal{H}^m \llcorner E$ which, by definition, satisfy the area formula

$$\int \operatorname{card}\{x \in A\colon P_V(x) = y\}\, d\mathcal{H}^m y = \int_A \det(P_V|T_E(x))\, d\mu_E x$$

for Borel sets $A \subset E$ and m-planes V. For rectifiable sets in finite dimensions, the planes $T_E(x)$ are just the approximate tangent planes. Then any set A with $\mathcal{H}^m(A) < \infty$ splits into m-pseudorectifiable and purely m-unpseudorectifiable parts. The latter is now defined by the property that it is a countable union of sets such that for each of them there exists a finite-dimensional linear subspace V such that all projections into finite-dimensional linear subspaces containing V

are purely m-unrectifiable, in the classical sense. With these notions the above procedure can be followed, but with notable complications.

Käenmäki, Sahlsten and Shmerkin [272] used ergodic-theoretic methods to investigate geometric properties of very general measures, involving also rectifiability, see also the survey [271]. Each tangent measure (recall Section 4.3) of a measure μ at x tells us something about μ around x only at scales that generate it, which can be very sparse. In order to have a more complete picture we should look at all tangent measures at x. In between there is a way to look at average behaviour via scenery flows and tangent distributions. These have been studied by many people, mainly in connection with fractal properties such as various dimensions.

A tangent distribution is a measure on a space of measures. Given a measure $\mu \in \mathcal{M}(B^n(0,1))$, define for $x \in \operatorname{spt}\mu$ and $t \geq 0$ the probability measure $\mu_{x,t}$ on $B^n(0,1)$ by

$$\mu_{x,t}(A) = \frac{\mu(e^{-t}A + x)}{\mu(B(x, e^{-t}))},\ A \subset B^n(0,1).$$

Then $(\mu_{x,t})_{t\geq 0}$ is called the scenery flow of μ at x. Letting δ_a denote the Dirac measure at a set for $T > 0$,

$$\langle\mu\rangle_{x,T} = \frac{1}{T}\int_0^T \delta_{\mu_{x,t}}\,dt.$$

Then any weak limit P of a sequence $\langle\mu\rangle_{x,T_i}, T_i \to \infty$ is called a tangent distribution of μ at x, $P \in \mathcal{TD}(\mu, x)$. They are probability measures on $\mathcal{M}(B^n(0,1))$ and they enjoy very strong translation and scaling invariance properties by a result of Hochman. The support of a tangential distribution at x is contained in the set of the restrictions to the unit ball of the tangent measures at x.

The authors of [272] studied in particular conical density and porosity properties. By Theorem 3.7, and its higher-dimensional analogues, for a purely m-unrectifiable set there is much measure in small cones around $(n-m)$-planes. The same is true for sets of Hausdorff dimension bigger than m. In [272], similar results are proven for general measures. In particular, the authors introduced a concept of average unrectifiability. A special case of their results states that if $\mu \in \mathcal{M}(B^n(0,1)), 0 \leq p < 1$, and for every $P \in \mathcal{TD}(\mu, x)$,

$$P(\{\nu \in \mathcal{M}(B^n(0,1))\colon \ \operatorname{spt}\nu \text{ is not } m\text{–rectifiable}\}) > p,$$

then for every $0 < s < 1$ there exists $0 < \varepsilon < 1$ such that

$$\liminf_{T\to\infty} \frac{1}{T}\mathcal{L}^1\left(\left\{t \in [0,T]\colon \inf_{V\in G(n,n-m)} \frac{\mu(X(x,V,s) \cap B(x,,e^{-t}))}{\mu(B(x,e^{-t}))} > \varepsilon\right\}\right) > p$$

for μ almost all $x \in \mathbb{R}^n$. A converse holds if μ has positive lower and finite upper density almost everywhere.

Hovila, Järvenpää and Ledrappier [242] proved that on a class of Riemann surfaces the union of complete geodesics has Hausdorff dimension 2 and $\mathcal{H}^2$ measure zero. To prove the second statement, they used their generalization of the Besicovitch–Federer projection theorem for transversal families, recall Section 4.5. The useful family was produced by investigating the geodesic flow on the tangent bundle.

Fuhrmann and Wang [217] proved that ergodic measures of certain dynamical systems on the 2-torus are 1-rectifiable.

16.3 Higher-Order Rectifiability

Anzellotti and Serapioni [26] introduced higher-order rectifiable sets. Let us say that $E \subset \mathbb{R}^n$ is (m, k, α)-rectifiable if there are m-dimensional $C^{k,\alpha}$ submanifolds M_i of $\mathbb{R}^n$ such that $\mathcal{H}^m(E \setminus \bigcup_i M_i) = 0$. Here k and m are positive integers and $0 \le \alpha \le 1$, with $C^{k,0}$ meaning C^k and $(m, k) = (m, k, 0)$. Of course, $(m, 1)$-rectifiable is then the same as m-rectifiable. Then, among m-rectifiable sets, Anzellotti and Serapioni characterized $(m, 2)$- and $(m, 1, \alpha)$-rectifiable sets in terms of non-homogeneous blow-ups. That is, the blow-up maps are rotations of $(x, y) \mapsto (r^{-1}x, r^{-1-\alpha}y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{n-m}$.

As observed in [26], the $(m, k, 1)$-rectifiable sets are the same as the $(m, k+1)$-rectifiable sets due to the Lusin-type theorem [203, Theorem 3.1.15]. The question of whether the upper limit in that theorem can be replaced by the approximate upper limit was solved in the negative by Kohn in [279].

Santilli [391] characterized all (m, k, α)-rectifiable sets with approximate differentiability. This means that he introduced a notion of approximate differentiability of higher order for subsets of $\mathbb{R}^n$ in analogy to the corresponding notion of functions (see [203, Section 3.1] and [252]) and he proved that an $\mathcal{H}^m$ measurable set with finite $\mathcal{H}^m$ measure is (m, k, α)-rectifiable if and only if it is almost everywhere approximately differentiable with parameters m, k, α. Santilli's definition of approximate differentiability of E at a involves conditions like

$$\lim_{r\to 0} r^{-m}\mathcal{H}^m\left(\left\{x \in E \cap B(a, r) : d(x, G) > sr^{k+\alpha}\right\}\right) = 0,$$

where G is a graph of a polynomial of degree at most k over an m-plane. We omit the precise definition, but we state the special case $k = 1$ more explicitly: E is

$(m, 1, \alpha)$-rectifiable if and only if for $\mathcal{H}^m$ almost all $a \in E$ there exist $V \in G(n, m)$ and $s > 0$ such that

$$\lim_{r \to 0} r^{-m} \mathcal{H}^m \left(\left\{ x \in E \cap B(a, r) \colon |P_{V^\perp}(x - a)| > s |P_V(x - a)|^{1+\alpha} \right\} \right) = 0. \quad (16.1)$$

Del Nin and Idu [170] used this formulation to give a different proof and a slightly more general result in this case, such as Theorem 3.7. When $\alpha > 0$, they also proved an analogue of Theorem 4.9. That is, assuming positive lower density, they gave a sufficient condition in terms of 'rotating' planes. The proof is much simpler than that of Theorem 4.9 since (16.1) forces the approximating planes to converge at a geometric rate. In [249], Idu and Maiale characterized in $\mathbb{H}^n$ the $(m, 1, \alpha)$ rectifiability, $n + 2 \leq m \leq 2n + 1$, in terms of approximate tangent paraboloids, following the scheme of [170].

Recently, there have been many other results related to higher-order rectifiability. Here are some.

Menne [338] defined higher-order differentiability of a set $A \subset \mathbb{R}^n$ by the approximability of $d(x, A)$ by polynomials over m-planes. He proved the higher-order rectifiability of the sets where this happens. In [339], he proved analogous results for distributions.

It is easy to see that for any subset E of $\mathbb{R}^n$ the set of points in E that can be touched by a ball from the complement is $(n - 1)$-rectifiable. Menne and Santilli [340] showed that the set where a closed subset of $\mathbb{R}^n$ can be touched by a ball from at least $n - m$ linearly independent directions is $(m, 2)$-rectifiable. The authors informed me that this also follows from Zajicek's results in [446]. Hajlasz had related results in [231].

Menne [337] proved the $(m, 2)$-rectifiability of integer multiplicity rectifiable varifolds whose first variation is a Radon measure, and Santilli [392] proved the same for general rectifiable varifolds with some extra conditions.

Bojarski, Hajlasz and Strzelecki [69] proved the C^k rectifiability of level sets of kth-order Sobolev mappings.

Kolasinski [280] and Ghinassi and Goering [224] gave for higher-order rectifiability sufficient conditions in terms of a Menger-type curvature (recall (3.2)), and Ghinassi [223] and Del Nin and Idu [170] in terms of square functions (recall Chapter 6).

Delladio has had many results on higher-order rectifiability, for example, of sets related to differentiability properties of functions, see [168, 169] and the references given there.

16.4 Fractal Rectifiability

Let $0 < s < m$ and $E \subset \mathbb{R}^n$ with $0 < \mathcal{H}^s(E) < \infty$. Then we could consider E as (m, s)-rectifiable if $\mathcal{H}^s \llcorner E$ is m-rectifiable according to Definition 4.2. In [312], it was studied how much of the theory of m-rectifiable sets could be extended to this setting. The paper contains several fairly easy positive results and many counterexamples.

Another possibility for (m, s)-rectifiability for non-integral s would be to use s/m-Hölder maps from $\mathbb{R}^m$. Some fractal curves of positive and finite s-dimensional Hausdorff measure, for example, the von Koch snowflake, can be parametrized by $1/s$-Hölder maps from $\mathbb{R}$, and they would be rectifiable in this sense. On the other hand, many standard self-similar Cantor sets meet such Hölder curves in measure zero, see [313], and they would be purely unrectifiable. Investigation along these lines was made by Badger and Vellis in [52]. In [48], Badger, Naples and Vellis gave an analyst's travelling salesman-type condition which implies that a set is contained in a $(1/s)$-Hölder curve.