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MINIMAL SURFACES IN 3-DIMENSIONAL SOLVABLE LIE GROUPS II

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An integral representation formula in terms of the normal Gauss map for minimal surfaces in 3-dimensional solvable Lie groups with left invariant metric is obtained.

1. INTRODUCTION

In the previous paper [**3**], we obtained an integral representation formula for minimal surfaces in the 3-dimensional solvable Lie group:

$$G(\mu_1,\mu_2) = (\mathbb{R}^3(x^1,x^2,x^3), g_{(\mu_1,\mu_2)}),$$

with group structure

$$(x^1,x^2,x^3)\cdot(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3) = (x^1+e^{\mu_1 x^3}\widetilde{x}^1, x^2+e^{\mu_2 x^3}\widetilde{x}^2, x^3+\widetilde{x}^3)$$

and metric

$$g_{(\mu_1,\mu_2)} = e^{-2\mu_1 x^3}(dx^1)^2 + e^{-2\mu_2 x^3}(dx^2)^2 + (dx^3)^2.$$

This two-parameter family of solvable Lie groups contains the following particular examples: Euclidean 3-space $\mathbb{E}^3$, hyperbolic 3-space H^3 and Euclidean motion group $E(1,1)$. Moreover, $G(0,1)$ is isometric to the Riemannian direct product $H^2\times\mathbb{E}^1$ of hyperbolic 2-space and the real line $\mathbb{E}^1$.

In this paper, we investigate the normal Gauss maps for surfaces in $G(\mu_1,\mu_2)$ and reformulate the integral representation formula of [**3**] in terms of the normal Gauss map.

On the other hand, study of minimal surfaces in the reducible Riemannian symmetric space $H^2\times\mathbb{E}^1$ has been started very recently by Rosenberg and his collaborators. See [**9, 10**].

In a recent paper [**7**], Mercuri, Montaldo and Piu obtained an integral representation formula for minimal surfaces in $H^2\times\mathbb{E}^1$ ([**7**, Theorem 5.1]). Their formula coincides with our formula for $G(0,1)$. Thus our formula is a unification of Góes–Simões–Kokubu formula [**2, 5**] and Mercuri–Montaldo–Piu formula [**7**].

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2. Solvable Lie group

In this paper, we study the following two-parameter family of homogeneous Riemannian 3-manifolds;

$$\{(\mathbb{R}^3(x^1,x^2,x^3), g_{(\mu_1,\mu_2)}) \mid (\mu_1,\mu_2)\in\mathbb{R}^2\}, \tag{2.1}$$

where the metrics $g_{(\mu_1,\mu_2)}$ are defined by

$$g_{(\mu_1,\mu_2)} := e^{-2\mu_1 x^3}(dx^1)^2 + e^{-2\mu_2 x^3}(dx^2)^2 + (dx^3)^2. \tag{2.2}$$

Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is realised as the following solvable matrix Lie group:

$$G(\mu_1,\mu_2) = \left\{ \begin{pmatrix} 0 & e^{\mu_1 x^3} & 0 & x^1 \\ 0 & 0 & e^{\mu_2 x^3} & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, x^1,x^2,x^3\in\mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{g}(\mu_1,\mu_2)$ is given explicitly by

$$\mathfrak{g}(\mu_1,\mu_2) = \left\{ \begin{pmatrix} 0 & 0 & 0 & y^3 \\ 0 & \mu_1 y^3 & 0 & y^1 \\ 0 & 0 & \mu_2 y^3 & y^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, y^1,y^2,y^3\in\mathbb{R} \right\}. \tag{2.3}$$

Then we can take the following orthonormal basis $\{E_1,E_2,E_3\}$ of $\mathfrak{g}(\mu_1,\mu_2)$:

$$E_1 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, E_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, E_3 = \begin{pmatrix} 0&0&0&1\\0&\mu_1&0&0\\0&0&\mu_2&0\\0&0&0&0 \end{pmatrix}.$$

Then the commutation relation of $\mathfrak{g}$ is given by

$$[E_1,E_2]=0,\ [E_2,E_3]=-\mu_2 E_2,\ [E_3,E_1]=\mu_1 E_1.$$

Left-translating the basis $\{E_1,E_2,E_3\}$, we obtain the following orthonormal frame field:

$$e_1 = e^{\mu_1 x^3}\frac{\partial}{\partial x^1},\ e_2 = e^{\mu_2 x^3}\frac{\partial}{\partial x^2},\ e_3 = \frac{\partial}{\partial x^3}.$$

The Levi-Civita connection ∇ of $G(\mu_1,\mu_2)$ is described by

$$\begin{aligned} &\nabla_{e_1}e_1 = \mu_1 e_3,\ \nabla_{e_1}e_2 = 0,\ \nabla_{e_1}e_3 = -\mu_1 e_1,\\ &\nabla_{e_2}e_1 = 0,\ \nabla_{e_2}e_2 = \mu_2 e_3,\ \nabla_{e_2}e_3 = -\mu_2 e_2,\\ &\nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = 0. \end{aligned} \tag{2.4}$$

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EXAMPLE 2.1. (Euclidean 3-space) The Lie group $G(0,0)$ is isomorphic and isometric to the Euclidean 3-space $\mathbb{E}^3 = (\mathbb{R}^3, +)$.

EXAMPLE 2.2. (Hyperbolic 3-space) Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c,c)$ is a warped product model of the hyperbolic 3-space:

$$H^3(-c^2) = (\mathbb{R}^3(x^1, x^2, x^3), e^{-2cx^3}\{(dx^1)^2 + (dx^2)^2\} + (dx^3)^2).$$

This matrix group model $G(c,c)$ is used by Góes–Simões [**2**] and Kokubu [**5**].

EXAMPLE 2.3. (Riemannian product $H^2(-c^2) \times \mathbb{E}^1$) Take $(\mu_1, \mu_2) = (0, c)$ with $c \neq 0$. Then the resulting homogeneous space is $\mathbb{R}^3$ with metric:

$$(dx^1)^2 + e^{-2cx^3}(dx^2)^2 + (dx^3)^2.$$

Hence $G(0,c)$ is identified with the Riemannian direct product of the Euclidean line $\mathbb{E}^1(x^1)$ and the warped product model

$$(\mathbb{R}^2(x^2, x^3), e^{-2cx^3}(dx^2)^2 + (dx^3)^2)$$

of $H^2(-c^2)$. Thus $G(0,c)$ is identified with $\mathbb{E}^1 \times H^2(-c^2)$.

EXAMPLE 2.4. (Solvmanifold) The model space Sol of the 3-dimensional *solvegeometry* [**11**] is $G(1,-1)$. The Lie group $G(1,-1)$ is isomorphic to the Minkowski motion group

$$E(1,1) := \left\{ \begin{pmatrix} e^{x^3} & 0 & x^1 \\ 0 & e^{-x^3} & x^2 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x^1, x^2, x^3 \in \mathbb{R} \right\}.$$

The full isometry group is $G(1,-1)$ itself. The homogeneous space

$$G(1,-1) = G(1,-1)/\{\mathrm{E}\}$$

is the only proper simply connected generalised Riemannian symmetric space of dimension 3. Here E is the identity matrix.

REMARK 2.1. Let $H^2(y_1, y_2)$ be the upper half plane model of the hyperbolic 2-space of constant curvature -1:

$$H^2(y^1, y^2) = \left(\{(y^1, y^2) \in \mathbb{R}^2 \mid y^2 > 0\}, \{(dy^1)^2 + (dy^2)^2\}/(y^2)^2\right).$$

Consider the warped product $H^2(y^1, y^2) \times_{y^2} \mathbb{E}^1(y^3)$ with warped product metric

$$\frac{(dy^1)^2 + (dy^2)^2}{(y^2)^2} + (y^2)^2(dy^3)^2.$$

Then it is easy to verify that this warped product is isometric to $E(1,1)$. In fact, the mapping $(y^1, y^2, y^3) := (x^1, e^{x^3}, x^2)$ is an isometry from $E(1,1)$ onto $H^2(y^1, y^2) \times_{y^2} \mathbb{E}^1(y^3)$.

Kokubu showed that every product minimal surface in the Riemannian product $\mathbb{E}^3(y^1,y^2,y^3) = \mathbb{E}^2(y^1,y^2) \times \mathbb{E}^1(y^3)$ is minimal in the warped product $H^2(y^1,y^2) \times_{y^2} \mathbb{E}(y^3)$ (see [**4**, Example 3.1]).

In particular, every (totally geodesic) plane $ay^1+by^2+c=0$ in the Euclidean 3-space $\mathbb{E}^3(y^1,y^2,y^3)$ is also minimal in this warped product. These planes are totally geodesic in the warped product if and only if y^1 = constant. Hence we notice that every plane "x^1 = constant" in $G(1,-1)$ is a totally geodesic surface.

EXAMPLE 2.5. $(H^2 \times_{(y^2)^2} S^1)$ Take $(\mu_1,\mu_2) = (-2,1)$. Then the resulting homogeneous space is $\mathbb{R}^3$ with metric $e^{4x^3}(dx^1)^2 + e^{-2x^3}(dx^2)^2 + (dx^3)^2$. Under the coordinate transformation: $(y^1,y^2,y^3) := (x^2, e^{x^3}, x^1)$, this homogeneous space is represented as the warped product $H^2 \times_f \mathbb{E}$ with base

$$H^2 = \Big(\big\{(y^1,y^2) \in \mathbb{R}^2 \mid \ y^2>0\big\}, \big\{(dy^1)^2+(dy^2)^2\big\}/(y^2)^2\Big),$$

standard fibre $\mathbb{E}^1 = (\mathbb{R}(y^3), (dy^3)^2)$, and the warping function $f(y^1,y^2) = (y^2)^2$. This metric induces a Riemannian metric on the coset space $G(-2,1)/\Gamma(-2,1)$, where the discrete subgroup $\Gamma(-2,1)$ is $\big\{(2\pi n,0,0) \in G(-2,1) \mid n \in \mathbb{Z}\big\}$. Kokubu has shown that the catenoid in Euclidean 3-space $G(0,0)$ is naturally regarded as a minimal surface in $G(-2,1)/\Gamma(-2,1)$ ([**4**, Example 3.3]). Note that the helicoid $z = \tan^{-1}(y/x)$ in Euclidean 3-space is naturally regarded as a "rotational" minimal surface in $\widetilde{E}(2)/\Gamma$, where $\widetilde{E}(2)$ is the universal covering of the Euclidean motion group $E(2)$ with flat metric and Γ is the discrete subgroup defined by $\Gamma := \big\{(0,0,2\pi n)\ n \in \mathbb{Z}\big\}$. (See [**3**, p. 83].)

EXAMPLE 2.6. Let D be the distribution spanned by e_1 and e_2. Since $[e_1,e_2] = 0$, this distribution is involutive. Now let M be the maximal integral surface of D through a point (x_0^1, x_0^2, x_0^3). Then (2.4) implies that M is flat and of constant mean curvature $(\mu_1+\mu_2)/2$. Moreover, one can check that this maximal integral surface is the plane $x^3 = x_0^3$.

(1) If $(\mu_1,\mu_2) = (0,0)$ then M is a totally geodesic plane.

(2) If $\mu_1 = \mu_2 = c \neq 0$. Then M is a horosphere in the hyperbolic 3-space $H^3(-c^2)$.

(3) If $\mu_1 = -\mu_2 \neq 0$. Then M is a non-totally geodesic minimal surface.

REMARK 2.2. Let $\mathrm{Gr}_2(TG)$ the Grassmann bundle of 2-planes over the Lie group $G = G(\mu_1,\mu_2)$. Take a nonempty subset Σ of $\mathrm{Gr}_2(TG)$. A surface M in G is said to be an Σ-*surface* if all the tangent planes of M belong to Σ. The collection of Σ-surfaces is called the Σ-*geometry*. In particular, if Σ is an orbit of G-action on $\mathrm{Gr}_2(TG)$, then Σ-geometry is said to be of *orbit type*. Now we regard G as a homogeneous space $G/\{\mathrm{E}\}$. Then every G-orbit in $\mathrm{Gr}_2(TG)$ is a homogeneous subbundle of $\mathrm{Gr}_2(TG)$. Hence the orbit space is identified with the Grassmann manifold $\mathrm{Gr}_2\big(\mathfrak{g}(\mu_1,\mu_2)\big)$. Take a unit vector W

in the Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ and denote by Π_W the linear 2-plane in $\mathfrak{g}(\mu_1, \mu_2)$ orthogonal to W. Let $\mathcal{O}(W)$ the orbit containing Π_W. Recently, Kuwabara [**6**] investigated $\mathcal{O}(W)$-surfaces in $G(\mu_1, \mu_2)$ with $\mu_1 = -\mu_2 \neq 0$.

3. Integral representation formula

Here we recall the integral representation formula obtained in the previous paper [**3**].

Let M be a Riemann surface and $(\mathfrak{D}, z)$ be a simply connected coordinate region. The exterior derivative d is decomposed as

$$d = \partial + \overline{\partial},\ \partial = \frac{\partial}{\partial z} dz,\ \overline{\partial} = \frac{\partial}{\partial \overline{z}} d\overline{z},$$

with respect to the conformal structure of M. Take a triplet $\{\omega^1, \omega^2, \omega^3\}$ of (1,0)-forms which satisfies the following differential system:

$$\overline{\partial}\omega^i = \mu_i \overline{\omega^i} \wedge \omega^3,\ i = 1, 2; \tag{3.1}$$

$$\overline{\partial}\omega^3 = \mu_1 \omega^1 \wedge \overline{\omega^1} + \mu_2 \omega^2 \wedge \overline{\omega^2}. \tag{3.2}$$

Proposition 3.1. *Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to (3.1)-(3.2) on a simply connected coordinate region $\mathfrak{D}$. Then*

$$\varphi(z, \overline{z}) = 2 \int_{z_0}^{z} \mathrm{Re}\big(e^{\mu_1 x^3(z,\overline{z})} \cdot \omega^1, e^{\mu_2 x^3(z,\overline{z})} \cdot \omega^2, \omega^3\big)$$

is a harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$. Conversely, any harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$ can be represented in this form.

Equivalently, the resulting harmonic map $\varphi(z, \overline{z})$ is defined by the following data:

$$\omega^1 = e^{-\mu_1 x^3} x_z^1\, dz,\ \omega^2 = e^{-\mu_1 x^3} x_z^2\, dz,\ \omega^3 = x_z^3\, dz, \tag{3.3}$$

where the coefficient functions are solutions to

$$x_{z\overline{z}}^i - \mu_i(x_z^3 x_{\overline{z}}^i + x_{\overline{z}}^3 x_z^i) = 0,\ (i = 1, 2) \tag{3.4}$$

$$x_{z\overline{z}}^3 + \mu_1 e^{-2\mu_1 x^3} x_z^1 x_{\overline{z}}^1 + \mu_2 e^{-2\mu_2 x^3} x_z^2 x_{\overline{z}}^2 = 0. \tag{3.5}$$

Corollary 3.1. *Let $\{\omega^1, \omega^2, \omega^3\}$ be a solution to*

$$\overline{\partial}\omega^i = \mu_i \overline{\omega^i} \wedge \omega^3,\ i = 1, 2; \tag{3.6}$$

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = 0 \tag{3.7}$$

on a simply connected coordinate region $\mathfrak{D}$. Then

$$\varphi(z, \overline{z}) = 2 \int_{z_0}^{z} \mathrm{Re}\big(e^{\mu_1 x^3(z,\overline{z})} \cdot \omega^1, e^{\mu_2 x^3(z,\overline{z})} \cdot \omega^2, \omega^3\big)$$

is a weakly conformal harmonic map of $\mathfrak{D}$ *into* $G(\mu_1, \mu_2)$. *Moreover* $\varphi(z, \bar{z})$ *is a minimal immersion if and only if*

$$\omega^1 \otimes \overline{\omega^1} + \omega^2 \otimes \overline{\omega^2} + \omega^3 \otimes \overline{\omega^3} \neq 0.$$

In particular for the product space $\mathbb{E}^1 \times H^2$, we have the following result.

COROLLARY 3.2. *Let* $\{\omega^1, \omega^2, \omega^3\}$ *be a solution to*

$$\bar{\partial}\omega^1 = 0, \quad \bar{\partial}\omega^2 = c\overline{\omega^2} \wedge \omega^3; \tag{3.8}$$

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = 0 \tag{3.9}$$

on a simply connected coordinate region $\mathfrak{D}$. *Then*

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^{z} \mathrm{Re}\big(\omega^1, e^{cx^3(z,\bar{z})} \cdot \omega^2, \omega^3\big) \tag{3.10}$$

is a weakly conformal harmonic map of $\mathfrak{D}$ *into the product space* $G(0, c)$. *Moreover* $\varphi(z, \bar{z})$ *is a minimal immersion if and only if*

$$\omega^1 \otimes \overline{\omega^1} + \omega^2 \otimes \overline{\omega^2} + \omega^3 \otimes \overline{\omega^3} \neq 0. \tag{3.11}$$

REMARK 3.1. The representation formula for minimal surfaces in $G(0, 1) = \mathbb{E}^1 \times H^2$ obtained by Mercuri–Montaldo–Piu [**7**, Theorem 5.1] coincides with (3.10). In [**7**], they assumed that the data $(\omega^1, \omega^2, \omega^3)$ satisfies (3.8), (3.9), (3.11) and the equation:

$$\bar{\partial}\omega^3 = \omega^2 \wedge \overline{\omega^2}. \tag{3.12}$$

However the equations (3.8)–(3.9) imply (3.11)–(3.12) under the assumption: there are no points on $\mathfrak{D}$ on which both ω^3 and $\bar{\partial}\omega^3$ vanish (*see* [**5**, Lemma 4.5]).

4. THE NORMAL GAUSS MAP

Let $\varphi : M \to G(\mu_1, \mu_2)$ be a conformal immersion. Take a unit normal vector field N along φ. Then, by the left translation we obtain the following smooth map:

$$\psi := \varphi^{-1} \cdot N : M \to S^2 \subset \mathfrak{g}(\mu_1, \mu_2).$$

The resulting map ψ takes value in the unit sphere in the Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$. Here, via the orthonormal basis $\{E_1, E_2, E_3\}$, we identify $\mathfrak{g}(\mu_1, \mu_2)$ with Euclidean 3-space $\mathbb{E}^3(u^1, u^2, u^3)$.

The smooth map ψ is called the *normal Gauss map* of φ.

Let $\varphi : \mathfrak{D} \to G(\mu_1, \mu_2)$ be a weakly conformal harmonic map of a simply connected Riemann surface $\mathfrak{D}$ determined by the data $(\omega^1, \omega^2, \omega^3)$. Express the data as $\omega^i = \phi^i \, dz$. Then the induced metric I of φ is

$$I = 2\left(\sum_{i=1}^{3} |\phi^i|^2\right) dz d\bar{z}.$$

Moreover these three coefficient functions satisfy

$$\frac{\partial \phi^3}{\partial \bar{z}} = -\sum_{i=1}^{2} \mu_i |\phi^i|^2, \quad \frac{\partial \phi^i}{\partial \bar{z}} = \mu_i \overline{\phi^i} \phi^3, \ i = 1, 2,$$

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0. \tag{4.1}$$

The harmonic map φ is a minimal immersion if and only if

$$|\phi^1|^2 + |\phi^2|^2 + |\phi^3|^2 \neq 0. \tag{4.2}$$

Here we would like to remark that ϕ^3 is identically zero if and only if φ is a plane $x^3 =$ constant. (See Example 2.6.) As we saw in Example 2.6, φ is minimal if and only if $\mu_1 + \mu_2 = 0$.

Hereafter we assume that ϕ^3 is not identically zero. Then we can introduce two mappings f and g by

$$f := \phi^1 - \sqrt{-1}\phi^2, \quad g := \frac{\phi^3}{\phi^1 - \sqrt{-1}\phi^2}. \tag{4.3}$$

By definition, f and g take values in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Using these two $\overline{\mathbb{C}}$-valued functions, φ is rewritten as

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^{z} \mathrm{Re}\Big(e^{\mu_1 x^3} \frac{1}{2} f(1 - g^2), e^{\mu_2 x^3} \frac{\sqrt{-1}}{2} f(1 + g^2), fg \Big)\, dz.$$

The normal Gauss map is computed as

$$\psi(z, \bar{z}) = \frac{1}{1 + |g|^2} \Big(2\,\mathrm{Re}(g) E_1 + 2\,\mathrm{Im}(g) E_2 + \big(|g|^2 - 1\big) E_3 \Big).$$

Under the stereographic projection $\mathcal{P} : S^2 \backslash \{\infty\} \subset \mathfrak{g}(\mu_1, \mu_2) \to \mathbb{C} := \mathbb{R}E_1 + \mathbb{R}E_2$, the map ψ is identified with the $\overline{\mathbb{C}}$-valued function g. Based on this fundamental observation, we call the function g the *normal Gauss map* of φ. The harmonicity together with the integrability (3.4)–(3.5) are equivalent to the following system for f and g:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} |f|^2 g \{ \mu_1 (1 - \bar{g}^2) - \mu_2 (1 + \bar{g}^2) \}, \tag{4.4}$$

$$\frac{\partial g}{\partial \bar{z}} = -\frac{1}{4} \{ \mu_1 (1 + g^2)(1 - \bar{g}^2) + \mu_2 (1 - g^2)(1 + \bar{g}^2) \} \bar{f}. \tag{4.5}$$

THEOREM 4.1. *Let f and g be a $\overline{\mathbb{C}}$-valued functions which are solutions to the system: (4.4)–(4.5). Then*

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^{z} \mathrm{Re}\big(e^{\mu_1 x^3} \frac{1}{2} f(1 - g^2), e^{\mu_2 x^3} \frac{\sqrt{-1}}{2} f(1 + g^2), fg \big)\, dz \tag{4.6}$$

is a weakly conformal harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2)$.

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PROOF: Since the harmonicity together with integrability is equivalent to (4.4)–(4.5), Proposition 3.1 implies the result. ∎

EXAMPLE 4.1. For the space form $G(c,c)$ of curvature $-c^2$, (4.4)–(4.5) reduces to

$$\frac{\partial f}{\partial \bar{z}} = -c|f|^2|g|^2\bar{g}, \quad \frac{\partial g}{\partial \bar{z}} = -\frac{c}{2}\bar{f}(1-|g|^4).$$

In particular, for Euclidean 3-space, we have

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} = 0.$$

In the case of hyperbolic 3-space $H^3(-c^2)$, one can deduce that g is a solution to the partial differential equation:

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} + \frac{2|g|^2\bar{g}}{1-|g|^4}\frac{\partial g}{\partial z}\frac{\partial g}{\partial \bar{z}} = 0. \tag{4.7}$$

The equation (4.7) means that g is a harmonic map into the extended complex plane $\overline{\mathbb{C}}(w)$ with singular metric (so-called *Kokubu metric*) $dwd\bar{w}/(1-|w|^4)$.

EXAMPLE 4.2. For $G(1,-1) = E(1,1)$, (4.4)–(4.5) reduces to

$$\frac{\partial f}{\partial \bar{z}} = |f|^2 g, \quad \frac{\partial g}{\partial \bar{z}} = -\frac{1}{2}(g+\bar{g})(g-\bar{g})\bar{f}.$$

EXAMPLE 4.3. For $G(0,c)$, f and g are solutions to

$$\frac{\partial f}{\partial \bar{z}} = -\frac{c}{2}|f|^2(1+\bar{g}^2), \quad \frac{\partial g}{\partial \bar{z}} = -\frac{c}{4}(1-g^2)(1+\bar{g}^2)\bar{f}.$$

EXAMPLE 4.4. Assume that $\mu_1 \neq 0$. Take the following two $\overline{\mathbb{C}}$-valued functions:

$$f = \frac{\sqrt{-1}}{\mu_1(z+\bar{z})}, \quad g = -\sqrt{-1}.$$

Then f and g are solutions to (4.4)–(4.5). By the integral representation formula, we can see that the minimal surface determined by the data (f,g) is a plane $x^2 = \text{constant}$. Note that this plane is totally geodesic in $G(1,-1)$.

EXAMPLE 4.5. Consider the product space $G(0,1)$, and take the following two functions f and g defined on $\mathbb{R}^2$.

$$\frac{\sqrt{-1}(f-1)}{2} = \frac{\tan(2y)(\cos(2x)+\sin(2y)) + \sqrt{-1}\sin(2x)}{2-\sin(2(x-y))+\sin(2(x+y))},$$

$$1-g^2 = \frac{2}{f}, \quad z = x+\sqrt{-1}y.$$

Then (f,g) is a solution to (4.4)–(4.5). Moreover it is easy to see that $\phi^1 = 1, (\phi^2)^2 + (\phi^3)^2 = -1$. One can check that the minimal surface determined by the data (f,g) is the *minimal helicoid* in the sense of Nelli and Rosenberg [8] (See also [7, Example 5.2]).

REMARK 4.1. In [**7**], the following two auxiliary functions were introduced.

$$\mathbf{G}^2 = \frac{f}{2}, \ \mathbf{H} = g \cdot \mathbf{G}.$$

Then we have

$$\phi^1 = \mathbf{G}^2 - \mathbf{H}^2, \ \phi^2 = \sqrt{-1}(\mathbf{G}^2 + \mathbf{H}^2), \ \phi^3 = 2\mathbf{G}\mathbf{H}.$$

These functions are solutions to the system:

$$\begin{aligned}\mathbf{G}_z &= \frac{\mathbf{H}}{2}\{\mu_1(\overline{\mathbf{G}}^2 - \overline{\mathbf{H}}^2) - \mu_2(\mathbf{G}^2 + \mathbf{H}^2)\},\\ \mathbf{H}_{\bar{z}} &= -\frac{1}{2\mathbf{G}}\{\mu_1(\mathbf{G}^2 + \mathbf{H}^2)(\overline{\mathbf{G}}^2 - \overline{\mathbf{H}}^2) + \mu_2(\mathbf{G}^2 - \mathbf{H}^2)(\overline{\mathbf{G}}^2 + \overline{\mathbf{H}}^2)\}\\ &\qquad + \frac{\mathbf{H}^2}{2\mathbf{G}}\{\mu_1(\overline{\mathbf{G}}^2 - \overline{\mathbf{H}}^2) - \mu_2(\mathbf{G}^2 + \mathbf{H}^2)\}.\end{aligned}$$

The integral representation formula is rewritten as

$$\varphi(z, \bar{z}) = 2\int_{z_0}^{z} \mathrm{Re}\big(e^{\mu_1 x^3}(\mathbf{G}^2 - \mathbf{H}^2), \sqrt{-1}e^{\mu_2 x^3}(\mathbf{G}^2 + \mathbf{H}^2), 2\mathbf{G}\mathbf{H}\big)\, dz.$$

In [**1**], Berdinskiĭ and Taĭmanov obtained a Weierstrass type representaion for minimal surfaces in Sol in terms of spinors and Dirac operators.

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