Introduction

There are two ways to think about $(\infty, 1)$-categories. The first is that an $(\infty, 1)$-category, as its name suggests, should be some kind of higher categorical structure. The second is that an $(\infty, 1)$-category should encode the data of a homotopy theory. So we first need to know what a homotopy theory is, and what a higher category is.

We can begin with the classical homotopy theory of topological spaces. In this setting, we consider topological spaces up to homotopy equivalence, or up to weak homotopy equivalence. Techniques were developed for defining a nice homotopy category of spaces, in which we define morphisms between spaces to be homotopy classes of maps between CW complex replacements of the original spaces being considered. However, the general framework here is not unique to topology; an analogous situation can be found in homological algebra. We can take projective replacements of chain complexes, then chain homotopy classes of maps, to define the derived category, the algebraic analogue of the homotopy category of spaces.

The question of when we can make this kind of construction (replacing by some particularly nice kinds of objects and then taking homotopy classes of maps) led to the definition of a model category by Quillen in the 1960s [100]. The essential information consists of some category of mathematical objects, together with some choice of which maps are to be designated as weak equivalences; these are the maps we would like to think of as invertible but may not be. The additional data of a model structure, and the axioms this data must satisfy, guarantee the existence of a well-behaved homotopy category as we have in the above examples, with no set-theoretic problems arising.

A more general notion of homotopy theory was developed by Dwyer and Kan in the 1980s. Their simplicial localization [57] and hammock localization [56] constructions provided a method in which a category with weak equivalences can be assigned to a simplicial category, or category enriched in

https://doi.org/10.1017/9781316181874.002 Published online by Cambridge University Press

simplicial sets. More remarkably, they showed that up to a natural notion of equivalence (now called Dwyer–Kan equivalence), every simplicial category arises in this way [55]. Thus, if a "homotopy theory" is just a category with weak equivalences, then we can think of simplicial categories as homotopy theories. In other words, simplicial categories provide a model for homotopy theories.

However, with Dwyer–Kan equivalences, the category of small simplicial categories itself forms a category with weak equivalences, and therefore has a homotopy theory. Hence, we have a "homotopy theory of homotopy theories". In fact, this category has a model structure, making it a homotopy theory in the more rigorous sense [27].

In practice, unfortunately, this model structure is not as nice as we might wish. It is not compatible with the monoidal structure on the category of simplicial categories, does not seem to have the structure of a simplicial model category in any natural way, and has weak equivalences which are difficult to identify for any given example. Therefore, a good homotopy theorist might seek an equivalent model structure with better properties.

An alternative model, that of complete Segal spaces, was proposed by Rezk [103]. Complete Segal spaces are simplicial diagrams of simplicial sets, satisfying some conditions which allow them to be thought of as something like simplicial categories but with weak composition. Their corresponding model category is cartesian, and is given by a localization of the Reedy model structure on simplicial spaces. Hence, the weak equivalences between fibrant objects are just levelwise weak equivalences of simplicial sets, and we have a good deal of extra structure that the model category of simplicial categories does not possess.

Meanwhile, in the world of category theory, simplicial categories were seen as models for $(\infty, 1)$-categories, or weak ∞-categories, with k-morphisms defined for all $k \geq 1$, that satisfy the property that, for $k > 1$, the k-morphisms are all weakly invertible. To see why simplicial categories provide a natural model, it is perhaps easier to consider instead topological categories, where we have a topological space of morphisms between any two objects. The 1-morphisms are just points in these mapping spaces. The 2-morphisms are paths between these points; at least up to homotopy, they are invertible. Then 3-morphisms are homotopies between paths, 4-morphisms are homotopies between homotopies, and we could continue indefinitely.

In the 1990s, Segal categories were developed as a weakened version of simplicial categories. They are simplicial spaces with discrete 0-space, and look like homotopy versions of the nerves of simplicial categories. They were first defined by Dwyer, Kan, and Smith [58], but developed from this categorical perspective by Hirschowitz and Simpson [70]. The model structure for Segal categories, begun in their work, was given explicitly by Pellissier [97].

Yet another model for $(\infty, 1)$-categories was given in the form of quasi-categories or weak Kan complexes, first defined by Boardman and Vogt [36]. They were developed extensively by Joyal, who defined many standard categorical notions, for example limits and colimits, within this more general setting. Although much of his work is still unpublished, the beginnings of these ideas can be found in [73]. The notion was adopted by Lurie, who established many of Joyal's results independently [88].

Finally, going back to the original motivation, Barwick and Kan proved that there is a model category on the category of small categories with weak equivalences; they instead use the term "relative categories" [11].

Comparisons between all these various models were conjectured by several people, including Toën [115] and Rezk [103]. In a slightly different direction, Toën proved that any model category satisfying a particular list of axioms must be Quillen equivalent to the complete Segal space model structure, hence axiomatizing what is meant to be a homotopy theory of homotopy theories, or homotopy theory of $(\infty, 1)$-categories [116].

Eventually, explicit comparisons were made, as shown in the following diagram:

$$\begin{tikzcd}
\mathcal{SC} \arrow[r, shift left] \arrow[drr, shift right] & \mathcal{S}e\mathcal{C}at_f \arrow[l, shift left] \arrow[r, shift left] & \mathcal{S}e\mathcal{C}at_c \arrow[l, shift left] \arrow[r, shift left] \arrow[d, leftrightarrow] & \mathcal{CSS} \arrow[l, shift left] \arrow[dl, leftrightarrow] \arrow[d, shift right] \\
 & & \mathcal{QC}at \arrow[ull, shift right] & \mathcal{R}el\mathcal{C}at \arrow[u, shift right]
\end{tikzcd}$$

The single arrows indicate that Quillen equivalences were given in both directions, and these were established by Joyal and Tierney [74]. The Quillen equivalence between simplicial categories and quasi-categories was proved in different ways by Joyal, Lurie [88], and Dugger and Spivak [51, 52]. The Quillen equivalence between complete Segal spaces and relative categories was given by Barwick and Kan [11]. The zigzag across the top row was established by the present author [30]. The original model structure for Segal categories is denoted by $\mathcal{S}e\mathcal{C}at_c$; the additional one $\mathcal{S}e\mathcal{C}at_f$ was established for the purposes of this proof.

In short, the purpose of this book is to make sense of this diagram. What, explicitly, are simplicial categories, Segal categories, quasi-categories, complete Segal spaces, and relative categories? What is the model category corresponding to each, and how can they be compared to one another? The answers to these questions have all been known and are in the literature, but we bring them together here.