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SEQUENCES OF WEAK SOLUTIONS FOR NON-LOCAL ELLIPTIC PROBLEMS WITH DIRICHLET BOUNDARY CONDITION

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Abstract In this paper the existence of infinitely many solutions for a class of Kirchhoff-type problems involving the p-Laplacian, with $p > 1$, is established. By using variational methods, we determine unbounded real intervals of parameters such that the problems treated admit either an unbounded sequence of weak solutions, provided that the nonlinearity has a suitable behaviour at ∞, or a pairwise distinct sequence of weak solutions that strongly converges to 0 if a similar behaviour occurs at 0. Some comparisons with several results in the literature are pointed out. The last part of the work is devoted to the autonomous elliptic Dirichlet problem.

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1. Introduction

In 1883 Kirchhoff proposed the relation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{K}$$

as an extension of the D'Alembert wave equation for free vibrations of elastic strings, where the above constants have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension (see [**24**]).

It is worth mentioning that (K) received much attention after the work of Lions [**30**], where a functional analysis framework was proposed for the problem. For instance, we refer the reader to [**3**, **13**, **19**] for some interesting results and further references. Recently, the study of the Kirchhoff equation has been considered in the elliptic case and involving the p-Laplacian operator.

Motivated by this interest, in this paper we deal with the following elliptic problem of Kirchhoff type:

$$\left.\begin{aligned} -\left(a+b\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)\Delta_p u+\alpha(x)|u|^{p-2}u=\lambda h(x)f(u) \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0, \end{aligned}\right\} \tag{K$_\lambda$}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, a and b are two non-negative constants (non-contemporarily zero), $p>1$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the usual p-Laplacian operator, $\alpha \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega}\alpha(x) \geqslant 0$, λ is a positive parameter, $f\colon \mathbb{R}\to\mathbb{R}$ is a continuous function and, finally, $h\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} h(x)>0$.

Many solvability conditions for Kirchhoff-type equations are given, such as the Yang index theory and invariant sets of descent flow (see [**37**, **41**]). However, for this kind of non-local problem there have been several multiplicity results using variational methods (see, for example, [**1**, **14**, **23**, **31**]).

Problem (K$_\lambda$) contains the following significant case:

$$\left.\begin{aligned} -\Delta_p u=\lambda f(u) \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0. \end{aligned}\right\} \tag{D$^f_\lambda$}$$

The existence of infinitely many solutions of the Dirichlet problem (D$^f_\lambda$) has been studied extensively. Most results assume that f is odd in order to apply some variant of the classical Lusternik–Schnirelmann theory. Only a few papers deal with nonlinearities having no symmetry properties. Among them, the ones that are closest to the present paper are certainly [**2**, **23**, **36**, **38**, **40**]. In particular, in [**36**], Omari and Zanolin proved that if

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^p}=0 \quad \text{and} \quad \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^p}=+\infty, \tag{1.1}$$

where

$$F(\xi):=\int_0^\xi f(t)\,\mathrm{d}t \quad (\xi\in\mathbb{R}),$$

(D$^f_\lambda$) has a sequence of non-zero and non-negative weak solutions, satisfying that $\max_{x\in\bar\Omega} u_n(x)\to 0$ as $n\to\infty$ (see also [**32**–**34**] and § 5).

In [**2**], Anello and Cordaro weakened condition (1.1) and obtained infinitely many positive solutions of (D$^f_\lambda$). The main idea of [**2**] is based on the general approach proposed by Ricceri [**38**], which yields weak solutions by searching for local minima of the underlying energy functional. This technique was suggested earlier in the paper of Saint Raymond [**40**]. Subsequently, following the cited approach, He and Zou [**23**] investigated the existence of infinitely many solutions for the problem

$$\left.\begin{aligned} -\left(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u=\lambda f(x,u) \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0, \end{aligned}\right\} \tag{H$_\lambda$}$$

where $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a suitable Carathéodory function (see Remark 4.7).

More recently, through the same variational approach, Dai and Liu studied the existence of infinitely many solutions for a non-local Kirchhoff-type equation involving the $p(x)$-Laplacian (see [**17**]). A similar analysis was also used by Dai and Wei [**18**] to investigate the existence of infinitely many solutions for a $p(x)$-Kirchhoff-type problem with Dirichlet boundary condition.

Here, inspired by the above-mentioned papers, we study the existence of infinitely many non-negative solutions to (K_λ). In practice, the previous circumstance is realized by showing that, under a suitable condition on the nonlinearity f, there exists a sequence of local minima $\{u_n\}$ for the functional associated with (K_λ).

More concretely, we determine intervals of parameters such that our problem admits either an unbounded sequence of solutions, provided that f has a suitable behaviour at ∞, or a pairwise distinct sequence of solutions that converges to 0 if a similar behaviour occurs at 0 (see Theorems 3.1 and 4.1, respectively). For instance, in Theorem 4.1, our key assumption at 0, along with the natural condition (k_1), can be formulated as the following algebraic inequality:

$$-\limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^p} < \delta^0_{\Omega,p}\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^p},$$

where

$$\delta^0_{\Omega,p} := \frac{\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}$$

is a real constant depending on the geometrical structure of Ω (see Remark 4.3).

For completeness, we mention that our results, for the Dirichlet case, are related to some recent contributions obtained by Kristály and Moroşanu in their interesting paper [**28**]. More precisely, they look for the existence of infinitely many non-negative solutions to the problem

$$\left.\begin{aligned} -\Delta u &= \lambda a(x)u^p + f(u) \quad \text{in } \Omega,\\ u|_{\partial\Omega} &= 0, \end{aligned}\right\} \tag{P_λ}$$

where $f\colon [0,+\infty[\,\to\mathbb{R}$ is a continuous function, the parameters p and λ are assumed to be positive and $a\in L^\infty(\Omega)$ is allowed to be indefinite. The crucial hypothesis adopted in the work is expressed by

$$-\infty < \liminf_{\xi\to L}\frac{F(\xi)}{\xi^2} \leqslant \limsup_{\xi\to L}\frac{F(\xi)}{\xi^2} = +\infty, \tag{1.2}$$

where either $L=0^+$ or $L=+\infty$. Moreover, a necessary preliminary approach is developed for the weight problem

$$\left.\begin{aligned} -\Delta u + K(x)u &= h(x,u) \quad \text{in } \Omega,\\ u|_{\partial\Omega} &= 0, \end{aligned}\right\} \tag{P^K_h}$$

where $K\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} K(x) > 0$ and $h\colon \Omega\times[0,+\infty[\,\to\mathbb{R}$ is a Carathéodory function satisfying certain properties (see also [**29**, Chapter 7]).

As with the above results, in Theorems 5.1 and 5.7, studying the unperturbed Dirichlet problem (D_λ^f) we require that

$$-\limsup_{\xi\to L}\frac{F(\xi)}{\xi^p} < \delta_{N,p}^L \liminf_{\xi\to L}\frac{F(\xi)}{\xi^p},$$

where, again, either $L = 0^+$ or $L = +\infty$ and

$$\delta_{N,p}^L := 2^{N+p}N\int_{1/2}^1 t^{N-1}(1-t)^p\,\mathrm{d}t;$$

see, for instance, Remark 5.6 for some details about the case when F possesses the above oscillating behaviour at 0.

As an example we present a particular existence result for a non-local elliptic problem defined on a Euclidean bounded domain $\Omega \subset \mathbb{R}^3$.

Theorem 1.1. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function such that*

$$\liminf_{\xi\to 0^+}\frac{F(\xi)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2} > 0.$$

Furthermore, assume that, for every $n\in\mathbb{N}$, there exist $\xi_n, \xi_n' \in \mathbb{R}$, with $0 < \xi_n < \xi_n'$ and $\lim_{n\to\infty}\xi_n' = 0$, such that $F(\xi_n) = \sup_{\xi\in[0,\xi_n']}F(\xi)$. There then exists $\lambda^* > 0$ such that, for every $\lambda > \lambda^*$, the problem

$$\left.\begin{aligned} -\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = \lambda f(u) \quad \textit{in } \Omega,\\ u|_{\partial\Omega} = 0 \end{aligned}\right\} \tag{N_λ}$$

admits a sequence $\{u_n\}$ of non-negative and non-trivial weak solutions strongly convergent to 0 in $H_0^1(\Omega)$ and such that $\lim_{n\to\infty}\|u_n\|_\infty = 0$.

We just observe that a more general condition than (1.1) in the low-dimensional case was introduced, very recently, by Bonanno and Molica Bisci [**4**], studying the existence of infinitely many weak solutions for a Sturm–Liouville problem. Subsequently, in [**5**], the same authors, by using this novel approach, studied (D_λ^f). There, (1.1) was replaced by the inequality

$$\liminf_{\xi\to 0^+}\frac{\max_{|t|\leqslant\xi}F(t)}{\xi^p} < \kappa\limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^p}, \tag{1.3}$$

where κ is a well-determined constant depending on the geometry of the open set Ω (see [**5**, Theorem 1] and Remark 5.6). This oscillating behaviour has been adopted for proving the existence of infinitely many weak solutions for different types of elliptic problems. Among others, we mention the works [**5**–**11**, **15**, **16**]. For a direct comparison with the above-mentioned results, with respect to (D_λ^f), see Remark 5.5.

The paper has the following structure. In § 2 we introduce our notation and the abstract Sobolev spaces setting. In §§ 3 and 4 we obtain our existence results (see Theorems 3.1

and 4.1) and some significant consequences, for instance, Corollaries 3.5 and 3.7, by using conditions on the nonlinearity f at ∞. Finally, § 5 is devoted to the autonomous Dirichlet problem (D^f_λ). To conclude, we cite the monographs [**20**] and [**29**] as general references on related topics.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^N$ (where $N > 1$) with smooth boundary $\partial\Omega$, $p > N/2$ and $h \in L^\infty(\Omega)$, such that $\operatorname{ess\,inf}_{x\in\Omega} h(x) > 0$. Furthermore, denote by X the space $W^{1,p}_0(\Omega)$ endowed by the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

We consider a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and define

$$F(\xi) := \int_0^\xi f(t)\,\mathrm{d}t \quad \text{for every } \xi \in \mathbb{R}.$$

In the case $N/2 < p \leqslant N$ we assume that f satisfies the following subcritical condition.

(h_∞) There exist $\delta \in \mathbb{R}^+$ and $q > 2p-1$, with $q < ((p-1)N+p)/(N-p)$ if $p < N$, such that

$$|f(t)| \leqslant \delta(1 + |t|^q)$$

for every $t \in \mathbb{R}$.

Moreover, let $J_\lambda\colon X \to \mathbb{R}$ be the energy functional associated with (K_λ) as

$$J_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \forall u \in X,$$

where

$$\Phi(u) := \frac{1}{p}\left(a\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x + \frac{b}{2}\left(\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x\right)^2 + \int_\Omega \alpha(x)|u(x)|^p\,\mathrm{d}x\right)$$

and

$$\Psi(u) := \int_\Omega h(x)F(u(x))\,\mathrm{d}x$$

for every $u \in X$.

It is well known that Φ is a continuously Gâteaux differentiable functional in X (at $u \in X$) whose derivative is given by

$$\Phi'(u)(v) := \left(a + b\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x\right)\int_\Omega |\nabla u|^{p-2}\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega \alpha(x)|u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x$$

for every $v \in X$. Furthermore, Φ is weakly lower semicontinuous and coercive.

On the other hand, standard arguments show that Ψ is a well-defined and continuously Gâteaux differentiable functional whose Gâteaux derivative (at $u \in X$) is given by

$$\Psi'(u)(v) := \int_\Omega h(x) f(u(x)) v(x) \, \mathrm{d}x$$

for every $v \in X$.

A function $u\colon \Omega \to \mathbb{R}$ is said to be a *weak solution* of (K_λ) if $u \in X$ and

$$\left(a + b \int_\Omega |\nabla u(x)|^p \, \mathrm{d}x\right) \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x + \int_\Omega \alpha(x) |u(x)|^{p-2} u(x) v(x) \, \mathrm{d}x - \lambda \int_\Omega h(x) f(u(x)) v(x) \, \mathrm{d}x = 0$$

for all $v \in X$. Hence, the critical points of J_λ are exactly the weak solutions of (K_λ).

Moreover, let

$$\tau := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega). \tag{2.1}$$

Simple calculations show that there exists $x_0 \in \Omega$ such that $B(x_0, \tau) \subset \Omega$, where $B(x_0, \tau)$ is the open ball of radius τ centred at the point x_0. We also define by

$$\omega_s := s^N \frac{\pi^{N/2}}{\Gamma(1 + N/2)}$$

the measure of the N-dimensional ball of radius $s > 0$, where Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} \mathrm{e}^{-z} \, \mathrm{d}z \quad \forall t > 0.$$

At this point, let $\theta \in X$ be the function

$$\theta(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \tau), \\ \dfrac{2}{\tau}(\tau - |x - x_0|) & \text{if } x \in B(x_0, \tau) \setminus B(x_0, \tau/2), \\ 1 & \text{if } x \in B(x_0, \tau/2), \end{cases}$$

which will be useful in the following, in the proof of our theorems. One has that

$$\|\theta\|^p = \int_\Omega |\nabla \theta(x)|^p \, \mathrm{d}x = \frac{2^p \omega_\tau}{\tau^p} \left(1 - \frac{1}{2^N}\right).$$

Indeed,

$$\begin{aligned} \int_\Omega |\nabla \theta(x)|^p \, \mathrm{d}x &= \int_{B(x_0,\tau) \setminus B(x_0,\tau/2)} \frac{2^p}{\tau^p} \, \mathrm{d}x \\ &= \frac{2^p}{\tau^p} (\operatorname{meas}(B(x_0, \tau)) - \operatorname{meas}(B(x_0, \tau/2))) \\ &= \frac{2^p \omega_\tau}{\tau^p} \left(1 - \frac{1}{2^N}\right). \end{aligned}$$

Finally, set

$$c_p := \left(\frac{\mathrm{meas}(\Omega)}{\omega_1}\right)^{1/N},$$

where 'meas(Ω)' stands for the Lebesgue measure of the open set Ω. As observed in [**22**, p. 157], the value c_p is the best constant that appears on the embedding $X \hookrightarrow L^p(\Omega)$.

3. Infinitely many non-negative solutions

In the result below, condition (h_1) states that the primitive of f must have an oscillating behaviour near to ∞. In this case we have the existence of a sequence of arbitrarily large weak solutions of problem (K_λ).

Theorem 3.1. *Let $b > 0$ and let $N/2 < p \leqslant N$. Furthermore, let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) \geqslant 0$. Assume that (h_∞) holds in addition to the following.*

(h_1) *For every $n \in \mathbb{N}$, there exist $\xi_n, \xi_n' \in \mathbb{R}$, with $0 \leqslant \xi_n < \xi_n'$ and $\lim_{n\to\infty} \xi_n = +\infty$, such that*

$$F(\xi_n) = \sup_{\xi\in[\xi_n,\xi_n']} F(\xi).$$

Furthermore, assume that there exists a real constant $\sigma_\infty > 0$ such that

(h_2)

$$\alpha_\infty := \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}} > -\sigma_\infty,$$

(h_3)

$$\beta_\infty := \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}} > \frac{\sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

Then, for every

$$\lambda > \frac{2^{2(p-N)}}{p\tau^{2p}}(a + \tfrac{1}{2}b + c_p^p\|\alpha\|_\infty) \times \frac{\omega_\tau^2(2^N-1)^2}{\beta_\infty \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x},$$

problem (K_λ) admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in X.

Proof. Fix λ as in the conclusion, define

$$g(x,t) := \begin{cases} h(x)f(t) & \text{if } t \geqslant 0,\\ h(x)f(0) & \text{if } t < 0 \end{cases}$$

for every $(x,t) \in \Omega \times \mathbb{R}$, and consider the problem

$$\left.\begin{aligned} -\left(a + b\int_\Omega |\nabla u|^p \, \mathrm{d}x\right)\Delta_p u + \alpha(x)|u|^{p-2}u = \lambda g(x,u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{aligned}\right\} \tag{K_λ^g}$$

Set

$$\Phi(u) := \frac{1}{p}\left[a\|u\|^p + \frac{b}{2}\|u\|^{2p} + \int_\Omega \alpha(x)|u(x)|^p \, \mathrm{d}x\right]$$

and

$$\Psi(u) := \int_\Omega \left(\int_0^{u(x)} g(x,t)\, \mathrm{d}t\right)\mathrm{d}x$$

for every $u \in X$. Abusing the notation, we denote here by Ψ the integral functional associated with our nonlinearity as well as with the truncated function g. The weak solutions of (K_λ^g) are the critical points of the functional

$$J_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \forall u \in X.$$

Owing to the compact embedding of X into $L^{q+1}(\Omega)$, the functional J_λ is well defined and sequentially weakly lower semicontinuous and continuously Gâteaux differentiable in X.

Now, fix $n \in \mathbb{N}$ and define

$$\mathbb{E}_n := \{u \in X : 0 \leqslant u(x) \leqslant \xi_n' \text{ almost everywhere (a.e.) in } \Omega\}.$$

Step 1. We can prove that the functional J_λ is bounded from below on $\mathbb{E}_n$ and that its infimum on $\mathbb{E}_n$ is attained at $u_n \in \mathbb{E}_n$.

Indeed, bearing in mind hypothesis (h_∞), clearly one has that

$$F(t) \leqslant \delta\left(|t| + \frac{|t|^{q+1}}{q+1}\right) \quad \forall t \in \mathbb{R}.$$

Hence, the inequality

$$\Psi(u) = \int_\Omega \left(\int_0^{u(x)} g(x,t)\, \mathrm{d}t\right)\mathrm{d}x \leqslant \delta\|h\|_\infty\left(\xi_n' + \frac{\xi_n'^{\,q+1}}{q+1}\right)\mathrm{meas}(\Omega)$$

holds for each $u \in \mathbb{E}_n$. Then,

$$\begin{aligned} J_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &\geqslant -\lambda\int_\Omega \left(\int_0^{u(x)} g(x,t)\, \mathrm{d}t\right)\mathrm{d}x \\ &\geqslant -\lambda\delta\|h\|_\infty\left(\xi_n' + \frac{\xi_n'^{\,q+1}}{q+1}\right)\mathrm{meas}(\Omega) \end{aligned}$$

for each $u \in \mathbb{E}_n$. Thus, J_λ is lower bounded in $\mathbb{E}_n$. It is clear that $\mathbb{E}_n$ is closed and convex, thus weakly closed in X. Let $\alpha_n := \inf_{u\in\mathbb{E}_n} J_\lambda(u)$. For every $k \in \mathbb{N}$, there exists $v_k \in \mathbb{E}_n$ such that

$$\alpha_n \leqslant J_\lambda(v_k) < \alpha_n + \frac{1}{k}.$$

Hence, it follows that

$$\begin{aligned} \Phi(v_k) &= \lambda\Psi(v_k) + J_\lambda(v_k) \\ &\leqslant \lambda\delta\|h\|_\infty\left(\xi_n' + \frac{{\xi_n'}^{q+1}}{q+1}\right)\operatorname{meas}(\Omega) + \alpha_n + \frac{1}{k} \\ &\leqslant \lambda\delta\|h\|_\infty\left(\xi_n' + \frac{{\xi_n'}^{q+1}}{q+1}\right)\operatorname{meas}(\Omega) + \alpha_n + 1. \end{aligned}$$

Then $\{v_k\}$ is a norm bounded in X. This implies that there exists a subsequence $\{v_{k_m}\}$ weakly convergent to $u_n \in \mathbb{E}_n$, being $\mathbb{E}_n$-weakly closed. At this point, we exploit the weak sequentially lower semicontinuity of J_λ and we obtain that $J_\lambda(u_n) = \alpha_n$.

Step 2. It follows that $u_n(x) \in [0, \xi_n]$ for almost every $x \in \Omega$.

In fact, fix $n \in \mathbb{N}$, define $h_n : \mathbb{R} \to \mathbb{R}$ as

$$h_n(t) = \begin{cases} \xi_n & \text{if } t > \xi_n, \\ t & \text{if } 0 \leqslant t \leqslant \xi_n, \\ 0 & \text{if } t < 0, \end{cases}$$

and consider the continuous superposition operator $T_n : X \to \mathbb{E}_n$,

$$T_n u(x) := h_n(u(x))$$

for every $u \in X$ and $x \in \Omega$. Moreover, one has that, for every $u \in X$, $T_n u \in \mathbb{E}_n$. We set $v_n^* = T_n u_n$ and

$$X_n := \{x \in \Omega : u_n(x) \notin [0, \xi_n]\}.$$

If $\operatorname{meas}(X_n) = 0$, our conclusion is achieved. Otherwise, suppose that $\operatorname{meas}(X_n) > 0$. Then, for almost every $x \in X_n$, one has that

$$\xi_n < u_n(x) \leqslant \xi_n',$$

as well as that

$$v_n^*(x) = T_n u_n(x) = \xi_n. \tag{3.1}$$

However, hypothesis (h_1) yields that

$$\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t \leqslant \sup_{t\in[\xi_n,\xi_n']} \int_0^t g(x,s)\,\mathrm{d}s = \int_0^{\xi_n} g(x,t)\,\mathrm{d}t = \int_0^{v_n^*(x)} g(x,t)\,\mathrm{d}t$$

for almost every $x \in X_n$. Hence,

$$\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t \leqslant \int_0^{v_n^*(x)} g(x,t)\,\mathrm{d}t \tag{3.2}$$

and $|\nabla v_n^*(x)| = 0$ for almost every $x \in X_n$. Hence, from (3.2), it follows that

$$\int_{X_n} \left(\int_{u_n(x)}^{v_n^*(x)} g(x,t)\,\mathrm{d}t \right) \mathrm{d}x \geqslant 0. \tag{3.3}$$

Furthermore, since $v_n^*(x) < |u_n(x)|$ for almost every $x \in X_n$, one has that

$$\int_{X_n} \alpha(x)(|u_n(x)|^p - |v_n^*(x)|^p)\,\mathrm{d}x \geqslant 0. \tag{3.4}$$

Then, by using (3.3) and (3.4), we easily get that

$$\begin{aligned}
&J_\lambda(v_n^*) - J_\lambda(u_n) \\
&\quad = \Phi(v_n^*) - \Phi(u_n) - \lambda \int_\Omega \left(\int_0^{v_n^*(x)} g(x,t)\,\mathrm{d}t \right) \mathrm{d}x + \lambda \int_\Omega \left(\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t \right) \mathrm{d}x \\
&\quad = -\frac{1}{p}\left[a \int_{X_n} |\nabla u_n(x)|^p\,\mathrm{d}x + \frac{b}{2}\left(\int_{X_n} |\nabla u_n(x)|^p\,\mathrm{d}x \right)^2 \right. \\
&\qquad\qquad\qquad \left. + \int_{X_n} \alpha(x)(|u_n(x)|^p - |v_n^*(x)|^p)\,\mathrm{d}x \right] \\
&\qquad\quad - \lambda \int_{X_n} \left(\int_{u_n(x)}^{v_n^*(x)} g(x,t)\,\mathrm{d}t \right) \mathrm{d}x \\
&\quad \leqslant -\frac{b}{2p}\left(\int_{X_n} |\nabla u_n(x)|^p\,\mathrm{d}x \right)^2.
\end{aligned}$$

Since $v_n^* \in \mathbb{E}_n$, it follows that $J_\lambda(v_n^*) \geqslant J_\lambda(u_n)$. Then

$$\int_{X_n} |\nabla u_n(x)|^p\,\mathrm{d}x = 0.$$

Whence we obtain

$$\|v_n^* - u_n\|^p = \int_\Omega |\nabla v_n^*(x) - \nabla u_n(x)|^p\,\mathrm{d}x = \int_{X_n} |\nabla u_n(x)|^p\,\mathrm{d}x = 0,$$

which means, since $\operatorname{meas}(X_n) > 0$, that $u_n(x) = v_n^*(x) \in [0, \xi_n]$ almost everywhere in Ω.

Step 3. We prove that u_n is a local minimum of J_λ in X.

To this end, let $u \in X$, let T_n be the operator defined above and let

$$X_n := \{x \in \Omega \colon u(x) \notin [0, \xi_n]\}.$$

Now, observe that

$$v_n^*(x) = T_n u(x) = \begin{cases} \xi_n & \text{if } u(x) > \xi_n, \\ u(x) & \text{if } 0 \leqslant u(x) \leqslant \xi_n, \\ 0 & \text{if } u(x) < 0. \end{cases} \tag{3.5}$$

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By the definition of the operator T_n, one has that

$$\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t = 0$$

if $x \in \Omega \setminus X_n$. Furthermore, if $x \in X_n$, then the following alternatives hold.

(a) If $u(x) < 0$, then

$$\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t = \int_0^{u(x)} g(x,t)\,\mathrm{d}t = \int_0^{u(x)} h(x)f(0)\,\mathrm{d}t = h(x)f(0)u(x) \leqslant 0.$$

(b) If $\xi_n < u(x) \leqslant \xi_n'$, then, by ($\mathrm{h}_1$), one has that

$$\begin{aligned}\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t &= \int_0^{u(x)} g(x,t)\,\mathrm{d}t - \int_0^{T_n u(x)} g(x,t)\,\mathrm{d}t \\ &= \int_0^{u(x)} g(x,t)\,\mathrm{d}t - \int_0^{\xi_n} g(x,t)\,\mathrm{d}t \\ &= \int_0^{u(x)} g(x,t)\,\mathrm{d}t - \sup_{t\in[\xi_n,\xi_n']} \int_0^t g(x,s)\,\mathrm{d}s \\ &\leqslant 0.\end{aligned}$$

(c) If $u(x) > \xi_n'$, we exploit (h_∞). Since $q > p-1$, it follows that

$$\begin{aligned}\int_{Tu(x)}^{u(x)} g(x,t)\,\mathrm{d}t &= \int_{\xi_n}^{u(x)} g(x,t)\,\mathrm{d}t \leqslant \delta \int_{\xi_n}^{u(x)} (1+t^q)\,\mathrm{d}t \\ &= \delta\biggl[(u(x)-\xi_n) + \frac{1}{q+1}(u(x)^{q+1} - \xi_n^{q+1})\biggr].\end{aligned}$$

Hence, the constant

$$C := \frac{\delta}{q+1} \sup_{\xi\geqslant\xi_n'} \biggl(\frac{(q+1)(\xi-\xi_n) + (\xi^{q+1} - \xi_n^{q+1})}{(\xi-\xi_n)^{q+1}}\biggr)$$

is finite and we have that

$$\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t \leqslant C|u(x) - T_n u(x)|^{q+1}$$

almost everywhere in Ω. We can then write that

$$\int_\Omega \biggl(\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x \leqslant C\gamma^{q+1}\|u - T_n u\|^{q+1},$$

where

$$\gamma := \sup_{u\in X\setminus\{0\}} \frac{(\int_\Omega |u(x)|^{q+1}\,\mathrm{d}x)^{1/(q+1)}}{\|u\|} < +\infty.$$

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Taking into account the above computations, for every $u \in X$, one has

$$\begin{aligned} J_\lambda(u) - J_\lambda(T_n u) &= \frac{1}{p}\bigg[a(\|u\|^p - \|T_n u\|^p) + \frac{b}{2}(\|u\|^{2p} - \|T_n u\|^{2p}) \\ &\qquad + \int_\Omega \alpha(x)(|u(x)|^p - |T_n u(x)|^p)\,\mathrm{d}x\bigg] \\ &\quad - \lambda \int_\Omega \bigg(\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t\bigg)\,\mathrm{d}x \\ &\geqslant \frac{a}{p}\int_{X_n} |\nabla u(x)|^p\,\mathrm{d}x + \frac{b}{2p}\bigg(\int_{X_n} |\nabla u(x)|^p\,\mathrm{d}x\bigg)^2 \\ &\quad - \lambda \int_\Omega \bigg(\int_{T_n u(x)}^{u(x)} g(x,t)\,\mathrm{d}t\bigg)\,\mathrm{d}x \\ &= \frac{a}{p}\int_\Omega |\nabla(u - T_n u)(x)|^p\,\mathrm{d}x + \frac{b}{2p}\bigg(\int_\Omega |\nabla(u - T_n u)(x)|^p\,\mathrm{d}x\bigg)^2 \\ &\quad - \lambda \int_\Omega \bigg(\int_{T u(x)}^{u(x)} g(x,t)\,\mathrm{d}t\bigg)\,\mathrm{d}x \\ &\geqslant \frac{a}{p}\|u - T_n u\|^p + \frac{b}{2p}\|u - T_n u\|^{2p} - C\gamma^{q+1}\lambda\|u - T_n u\|^{q+1}. \end{aligned}$$

Since $T_n u \in \mathbb{E}_n$, it follows that $J_\lambda(T_n u) \geqslant J_\lambda(u_n)$. We then have

$$J_\lambda(u) \geqslant J_\lambda(u_n) + \|u - T_n u\|^{2p}\bigg(\frac{b}{2p} - C\gamma^{q+1}\lambda\|u - T_n u\|^{q+1-2p}\bigg).$$

Moreover, since T_n is continuous in X (see [**32**]), $u_n = T_n u_n$, $q + 1 - 2p > 0$ and

$$\|u - T_n u\| \leqslant \|u - u_n\| + \|u_n - T_n u\| = \|u - u_n\| + \|T_n u_n - T_n u\|,$$

there exists $\beta > 0$ such that

$$\|u - T_n u\|^{q+1-2p} \leqslant \frac{b}{4p\lambda C\gamma^{q+1}}$$

for every $u \in X$ with $\|u - u_n\| < \beta$. Hence, if $\|u - u_n\| < \beta$, it follows that

$$J_\lambda(u) \geqslant J_\lambda(u_n) + \frac{b}{4p}\|u - T_n u\|^{2p} \geqslant J_\lambda(u_n),$$

that is, u_n is a local minimum of J_λ in X.

Step 4. We prove that $\liminf_{n\to\infty} \alpha_n = -\infty$.
Exploiting (h_2), there exists $\varrho > 0$ such that

$$F(\xi) > -\sigma_\infty \xi^{2p}$$

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for every $\xi > \varrho$. Furthermore, let $\{\eta_k\} \subset]0, +\infty[$ be a sequence such that $\lim_{k\to\infty} \eta_k = +\infty$ and

$$\lim_{k\to\infty} \frac{F(\eta_k)}{\eta_k^{2p}} = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}}. \tag{3.6}$$

We can choose a subsequence $\{\xi'_{n_k}\}$ of $\{\xi'_n\}$ such that $\xi'_{n_k} \geqslant \eta_k$ for every $k \in \mathbb{N}$. Thus, the function $\theta_k := \eta_k\theta$ belongs to $\mathbb{E}_{n_k}$ for every $k \in \mathbb{N}$. Now, observe that

$$\Phi(\theta_k) \leqslant \frac{\eta_k^{2p}}{p}\left(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\right)\|\theta\|^{2p} \tag{3.7}$$

for every $k \geqslant k_0$. One then has

$$J_\lambda(\theta_k) \leqslant \frac{\eta_k^{2p}}{p}\left(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\right)\|\theta\|^{2p} \\ - \lambda\left[F(\eta_k)\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x + \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)F(\theta_k(x))\,\mathrm{d}x\right]$$

for every $k \geqslant k_0$. Hence,

$$J_\lambda(\theta_k) \leqslant \frac{\eta_k^{2p}}{p}\left(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\right)\|\theta\|^{2p} \\ - \lambda\left[F(\eta_k)\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x + \int_{G_\varrho} h(x)F(\theta_k(x))\,\mathrm{d}x + \int_{G^\varrho} h(x)F(\theta_k(x))\,\mathrm{d}x\right],$$

where

$$G_\varrho := \{x \in B(x_0,\tau) \setminus B(x_0,\tau/2) \colon 0 \leqslant \theta_k(x) \leqslant \varrho\}$$

and

$$G^\varrho := \{x \in B(x_0,\tau) \setminus B(x_0,\tau/2) \colon \theta_k(x) > \varrho\}.$$

Now, by using the mean value theorem, we obtain

$$\left|\int_{G_\varrho} h(x)F(\theta_k(x))\,\mathrm{d}x\right| \leqslant \|h\|_\infty \operatorname{meas}(\Omega) \max_{t\in[0,\varrho]} |f(t)|\varrho. \tag{3.8}$$

Inequalities (3.7) and (3.8) then yield

$$J_\lambda(\theta_k) \leqslant \eta_k^{2p}\left[\left(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\right)\frac{\|\theta\|^{2p}}{p} \right. \\ - \lambda\left(\frac{F(\eta_k)}{\eta_k^{2p}}\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \frac{\|h\|_\infty \operatorname{meas}(\Omega)}{\eta_k^{2p}} \max_{t\in[0,\varrho]} |f(t)|\varrho \right. \\ \left.\left. - \sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x\right)\right].$$

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Thus, taking into account the choice of the parameter λ, the right-hand side goes to $-\infty$ as $k \to \infty$. Hence, clearly one has that $\lim_{k\to\infty} J_\lambda(\theta_k) = -\infty$. Moreover, since

$$\alpha_{n_k} := \inf_{u\in\mathbb{E}_{n_k}} J_\lambda(u) \leqslant J_\lambda(\theta_k),$$

the previous inequality implies that $\lim_{k\to\infty} \alpha_{n_k} = -\infty$.

At this point, we can prove that the sequence of local minima u_{n_k} must be unbounded. In fact, if it were bounded, there would be a subsequence, again denoted by u_{n_k}, weakly convergent to some function $\bar{u} \in X$. We then have the contradiction

$$J_\lambda(\bar{u}) \leqslant \liminf_{k\to\infty} J_\lambda(u_{n_k}) = -\infty,$$

and the assertion is completely proved. □

Remark 3.2. The assumptions adopted in our results are strictly related to some other theorems contained in [**29**, Chapter 7], as pointed out in § 1, where Kristály *et al.* studied the existence of infinitely many weak solutions for the Dirichlet problem (see, for instance, [**29**, Theorem 7.8]). In our case, due to the presence of the parameter λ, we are able to also treat elliptic Dirichlet problems in which (1.2) is violated. See § 5 for more details.

Remark 3.3. It is simple to see that (h_2) and (h_3) can be replaced by the following (equivalent) algebraic inequality:

$$-\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}} < \delta^{\infty}_{\Omega,2p} \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}}, \qquad (\mathrm{G}^\infty)$$

where we set

$$\delta^{\infty}_{\Omega,2p} := \frac{\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

From (G^∞) there exists $\sigma_\infty > 0$ such that, for every

$$\lambda > \frac{2^{2(p-N)}}{p\tau^{2p}}(a + \tfrac{1}{2}b + c_p^p\|\alpha\|_\infty) \times \frac{\omega_\tau^2(2^N-1)^2}{\beta_\infty \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x},$$

problem (K_λ) admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in X.

Remark 3.4. After a careful analysis of the above proof, the reader can observe that Theorem 3.1 also holds true in the low-dimensional case, i.e. $p > N$. In this setting our conclusion can be achieved without condition (h_∞), due to the presence of the compact embedding $X \hookrightarrow C^0(\bar{\Omega})$. Moreover, as is easy to see, if $a > 0$ in the higher-dimensional case, the growth condition can be relaxed and we can assume that $p > 1$.

From now on, in this section, we assume that $b > 0$ in addition to condition (h_∞) when the case $N/2 < p \leqslant N$ is exploited.

Corollary 3.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) \geqslant 0$ such that* (h_1) *holds in addition to*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^{2p}} = 0 \quad \textit{and} \quad \beta_\infty := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^{2p}} > 0.$$

There then exists $\sigma_\infty > 0$ such that, for every

$$\lambda > \frac{2^{2(p-N)}}{p\tau^{2p}}(a + \tfrac{1}{2}b + c_p^p \|\alpha\|_\infty) \times \frac{\omega_\tau^2 (2^N - 1)^2}{\beta_\infty \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x},$$

problem (K_λ) admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in X.

Proof. The result is an elementary consequence of Theorem 3.1. Indeed, since

$$\limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^{2p}} > 0,$$

one can fix $\sigma_\infty > 0$ such that

$$\sigma_\infty < \frac{\beta_\infty \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}{\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x}.$$

On the other hand,

$$0 = \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^{2p}} > -\sigma_\infty.$$

The proof is complete. □

Example 3.6. Let Ω be a smooth domain of $\mathbb{R}^3$ and consider the continuous function $j\colon \mathbb{R} \to \mathbb{R}$ defined by

$$j(t) := \begin{cases} t^4 \sin 2t + 4t^3 \sin^2 t & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0, \end{cases}$$

and whose potential is

$$J(\xi) = \int_0^\xi j(t)\,\mathrm{d}t = \xi^4 \sin^2 \xi.$$

It is elementary to prove that all the hypotheses of Corollary 3.5 are verified. Then, for every

$$\lambda > \frac{49}{8}\left(a + \frac{b}{2}\right) \frac{1}{\tau^4(\omega_{\tau/2} - \sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} \theta(x)^4\,\mathrm{d}x)},$$

the problem

$$-\left(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = \lambda j(u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0$$

admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in $H_0^1(\Omega)$.

In the next consequence we assume the stronger condition that

$$\lim_{\xi\to+\infty}\frac{F(\xi)}{\xi^{2p}}=+\infty.$$

Corollary 3.7. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function with $f(0)\geqslant 0$ such that* (h$_1$) *holds. Furthermore, assume that*

$$\lim_{\xi\to+\infty}\frac{F(\xi)}{\xi^{2p}}=+\infty.$$

Then, for every $\lambda>0$, (K_λ) admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in X.

Example 3.8. Let Ω be a smooth domain of $\mathbb{R}^3$ and consider the continuous function $k\colon\mathbb{R}\to\mathbb{R}$ defined by

$$k(t):=\begin{cases}t^4(1/2-\sin(t^{3/4})) & \text{if } t>0,\\ 0 & \text{if } t\leqslant 0.\end{cases}$$

An easy computation ensures that all the hypotheses of Corollary 3.7 are verified; in particular, we have that

$$\begin{aligned}\frac{\int_0^\xi k(t)\,\mathrm{d}t}{\xi^4}=&\frac{4(81\xi^3-2142\xi^{3/2}+20\,944)}{243\xi^{11/4}}\cos\xi^{3/4}\\ &-\frac{68(81\xi^3-1386\xi^{3/2}+6160)}{729\xi^{7/2}}\sin\xi^{3/4}\\ &+\frac{209\,440}{729\xi^4}\int_0^\xi\frac{\sin t^{3/4}}{\sqrt{t}}\,\mathrm{d}t+\frac{\xi}{10}\to+\infty\end{aligned}$$

as $\xi\to+\infty$. Then, for every $\lambda>0$, the problem

$$-\biggl(a+b\int_\Omega|\nabla u|^2\,\mathrm{d}x\biggr)\Delta u=\lambda k(u)\quad\text{in }\Omega,$$
$$u|_{\partial\Omega}=0$$

admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in $H_0^1(\Omega)$.

Finally, the following proposition is a consequence of Theorem 3.1.

Proposition 3.9. *Let $\{a_n\},\{b_n\}$ be two sequences in $]0,+\infty[$, $a_n<b_n<a_{n+1}$ (for all $n\geqslant n_0$, for some $n_0\in\mathbb{N}$), $\lim_{n\to\infty}b_n=+\infty$ and $\lim_{n\to\infty}b_n/a_n=+\infty$. Moreover, let $\varphi_1,\varphi_2\in C^1([0,1])$ be two non-negative and non-zero functions such that $\varphi_i(0)=\varphi_i(1)=\varphi_i'(0)=\varphi_i'(1)=0$ (for $i=1,2$), and define the function $r\colon\mathbb{R}\to\mathbb{R}$ as*

$$r(t):=\begin{cases}\varphi_1\biggl(\dfrac{t-b_n}{a_{n+1}-b_n}\biggr) & \textit{if } t\in\bigcup_{n\geqslant n_0}[b_n,a_{n+1}],\\ -\varphi_2\biggl(\dfrac{t-a_{n+1}}{b_{n+1}-a_{n+1}}\biggr) & \textit{if } t\in\bigcup_{n\geqslant n_0}\,]a_{n+1},b_{n+1}[,\\ 0 & \textit{otherwise.}\end{cases}$$

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Furthermore, let $p > N$ *and assume that there exists a constant* $\sigma_\infty > 0$ *such that* $\max_{s\in[0,1]} \varphi_2(s) < \sigma_\infty$ *and*

$$\max_{s\in[0,1]} \varphi_1(s) > \frac{\sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

Then, for every

$$\lambda > \frac{2^{2(p-N)}}{p\tau^{2p}}\left(a + \frac{b}{2}\right) \times \frac{\omega_\tau^2(2^N - 1)^2}{\max_{s\in[0,1]} \varphi_1(s) \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x},$$

the problem

$$\left.\begin{aligned} -\left(a + b\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)\Delta_p u &= \lambda h(x) y(u) \quad \text{in } \Omega,\\ u|_{\partial\Omega} &= 0, \end{aligned}\right\} \tag{G_λ}$$

where

$$y(u) := |u|^{2p-1}(2pr(u) + ur'(u)),$$

admits an unbounded sequence $\{u_n\}$ *of non-negative weak solutions in* X*.*

Proof. Let $\{a_n\}, \{b_n\}$ be two positive sequences satisfying our assumptions. We claim that all the hypotheses of Theorem 3.1 are verified. Indeed, one has that

$$F(\xi) := \int_0^\xi y(t)\,\mathrm{d}t = \xi^{2p} r(\xi) \quad \forall \xi \in \mathbb{R}^+.$$

Moreover, direct computations ensure that

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}} = \liminf_{\xi\to+\infty} r(\xi) = -\max_{s\in[0,1]} \varphi_2(s) > -\sigma_\infty$$

and

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}} = \limsup_{\xi\to+\infty} r(\xi) = \max_{s\in[0,1]} \varphi_1(s) > \frac{\sigma_\infty \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

Hence, for every parameter λ, as in the conclusion, Theorem 3.1 and Remark 3.3 guarantee the existence of an unbounded sequence of weak solutions of (G_λ). □

A concrete application of the above result is presented in the following.

Example 3.10. Let $\Omega \in \mathbb{R}^N$ be an open set of smooth boundary and let $h \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} h(x) > 0$. Furthermore, take

$$a_n := n! \quad \text{and} \quad b_n := n!n$$

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for every $n \geqslant 2$. Now, define $\varphi_1, \varphi_2 \in C^1([0,1])$ as

$$\varphi_1(s) := \alpha e^4 e^{1/s(s-1)}, \quad \varphi_2(s) := \beta e^4 e^{1/s(s-1)} \quad \forall s \in [0,1],$$

where $\beta > 0$ and

$$\alpha > \frac{(\beta+1)\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^4 \, dx}{\int_{B(x_0,\tau/2)} h(x)\, dx}.$$

Set $p > N$ and

$$r(t) := \begin{cases} \varphi_1\left(\dfrac{t - n!n}{(n+1)! - n!n}\right) & \text{if } t \in \bigcup_{n\geqslant 2} [n!n, (n+1)!], \\ -\varphi_2\left(\dfrac{t-(n+1)!}{(n+1)!(n+1)-(n+1)!}\right) & \text{if } t \in \bigcup_{n\geqslant 2}](n+1)!, (n+1)!(n+1)[, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every

$$\lambda > \frac{2^{2(p-N)}}{p\tau^{2p}}\left(a + \frac{b}{2}\right) \frac{\omega_\tau^2 (2^N - 1)^2}{\alpha \int_{B(x_0,\tau/2)} h(x)\, dx - (\beta+1)\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^{2p}\, dx},$$

the problem

$$-\left(a + b\int_\Omega |\nabla u|^p \, dx\right)\Delta_p u = \lambda h(x) y(u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

where

$$y(u) := |u|^{2p-1}(2pr(u) + ur'(u)),$$

admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in X.

Remark 3.11. Instead of the function θ used in our results, it is possible to work with a suitably assigned cut-off function $\vartheta \in C_0^\infty(\Omega)$, such that $\vartheta \in X$. More precisely, if we assume that there exists a compact subset $D \subset \operatorname{supp}(\vartheta)$, such that $0 \leqslant \vartheta(x) \leqslant 1$ for every $x \in \Omega$ and $\vartheta|_D = 1$, condition (h_3) in Theorem 3.1 can be replaced by the hypothesis

$$\beta_\infty := \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^{2p}} > \frac{\sigma_\infty \int_{\Omega\setminus D} h(x)\theta(x)^{2p}\, dx}{\int_D h(x)\, dx},$$

obtaining that, for every

$$\lambda > \frac{1}{p}\left(a + \frac{b}{2} + c_p^p \|\alpha\|_\infty\right) \frac{\|\vartheta\|}{\beta_\infty \int_D h(x)\, dx - \sigma_\infty \int_{\operatorname{supp}(\vartheta)\setminus D} h(x)\theta(x)^{2p}\, dx},$$

problem (K_λ) admits an unbounded sequence $\{u_n\}$ of non-negative weak solutions in X. A direct (and well-known) construction of ϑ is recalled in [**6**].

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4. Arbitrarily small non-negative solutions

By slightly modifying the assumptions in Theorem 3.1, we can also obtain the existence of a sequence of non-trivial arbitrarily small weak solutions. In particular, in this case, we require that the primitive of f has an oscillating behaviour near the origin expressed by condition (k_1). Assuming that $p > N/2$, the statements of our result are as follows.

Theorem 4.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, with $f(0) = 0$, satisfying the following condition.*

(k_1) *For every $n \in \mathbb{N}$, there exist $\xi_n, \xi_n' \in \mathbb{R}$, with $0 \leqslant \xi_n < \xi_n'$ and $\lim_{n\to\infty} \xi_n' = 0$, such that*

$$F(\xi_n) = \sup_{\xi \in [\xi_n, \xi_n']} F(\xi).$$

Furthermore, assume that there exists a real constant σ_0 such that

(k_2)

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} > -\sigma_0,$$

(k_3)

$$\beta_0 := \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} > \frac{\sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p \,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

Then, for every

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(a + \frac{b}{2} + c_p^p \|\alpha\|_\infty\right) \frac{\omega_\tau(2^N - 1)}{\beta_0 \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x},$$

problem (K_λ) admits a sequence $\{u_n\}$ of non-negative and non-trivial weak solutions strongly convergent to 0 in X and such that $\lim_{n\to\infty} \|u_n\|_\infty = 0$.

Proof. The first steps of our proof are similar to [**23**, Theorem 2.1]. For our purposes, we start by choosing

$$q \in \left]2p - 1, \frac{(p-1)N + p}{N - p}\right[$$

if $p < N$. In the other cases it is enough to choose $q > 2p - 1$. Furthermore, fix λ as in the conclusions and fix $\bar{t} > 0$. By our assumptions on the data, fixing $\bar{t} > 0$, there exists $\delta > 0$ such that, for every $0 \leqslant t \leqslant \bar{t}$ and almost every $x \in \Omega$, one has

$$|h(x)f(t)| \leqslant \delta.$$

Without loss of generality, we suppose that, for every $n \in \mathbb{N}$, $\max\{\xi_n', \xi_n\} \leqslant \bar{t}$. Let λ be as in the condition, and define $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$g(x,t) := \begin{cases} h(x)f(\bar{t}) & \text{if } t > \bar{t}, \\ h(x)f(t) & \text{if } 0 \leqslant t \leqslant \bar{t}, \\ 0 & \text{if } t < 0. \end{cases}$$

Whence, for almost every $x \in \Omega$ and $t \in \mathbb{R}$, it turns out that

$$|g(x,t)| \leqslant \delta. \tag{4.1}$$

Now, consider the problem (K_λ^g),

$$-\left(a + b\int_\Omega |\nabla u|^p \, \mathrm{d}x\right)\Delta_p u + \alpha(x)|u|^{p-2}u = \lambda g(x,u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

and set

$$J_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \forall u \in X,$$

where

$$\Phi(u) := \frac{1}{p}\left[a\|u\|^p + \frac{b}{2}\|u\|^{2p} + \int_\Omega \alpha(x)|u(x)|^p \, \mathrm{d}x\right]$$

and

$$\Psi(u) := \int_\Omega \left(\int_0^{u(x)} g(x,t)\,\mathrm{d}t\right)\mathrm{d}x$$

for every $u \in X$. Again abusing the notation, we denote here by Ψ the integral functional associated with the truncated map g. Clearly, the weak solutions of (K_λ^g) are the critical points of the energy functional J_λ.

Owing to (4.1) and the compact embedding of X into $L^{q+1}(\Omega)$ (respectively, into $C^0(\bar{\Omega})$ if $p > N$), the functional J_λ is well defined and sequentially weakly lower semi-continuous and continuously Gâteaux differentiable in X.

Taking into account (4.1) and (k_1) and using the same methods as applied in the proof of Theorem 3.1, one can prove that, for every $n \in \mathbb{N}$, J_λ admits a local minimum u_n that belongs to the set

$$\mathbb{E}_n := \{u \in X : 0 \leqslant u(x) \leqslant \xi_n' \text{ a.e. in } \Omega\}.$$

More precisely, every u_n assumes its values in the interval $[0, \xi_n]$ except for a null Lebesgue measure subset of Ω. Set $\alpha_n := \inf_{u\in\mathbb{E}_n} J_\lambda(u) = J_\lambda(u_n)$. For every $u \in \mathbb{E}_n$, by using (4.1), one has that

$$\begin{aligned} J_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &\geqslant -\lambda \int_\Omega \left(\int_0^{u(x)} g(x,t)\,\mathrm{d}t\right)\mathrm{d}x \\ &\geqslant -\delta\lambda \operatorname{meas}(\Omega)\xi_n'. \end{aligned}$$

Then, since $-\delta\lambda \operatorname{meas}(\Omega)\xi_n' \leqslant \alpha_n \leqslant 0$, it follows that

$$\lim_{n\to\infty} \alpha_n = 0. \tag{4.2}$$

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At this point we observe that

$$\begin{aligned}\Phi(u_n) &= \lambda\Psi(u_n) + J_\lambda(u_n)\\ &\leqslant \lambda \int_\Omega \biggl(\int_0^{u_n(x)} g(x,t)\,\mathrm{d}t\biggr)\,\mathrm{d}x + \alpha_n\\ &\leqslant \delta\lambda\,\mathrm{meas}(\Omega)\xi_n' + \alpha_n.\end{aligned}$$

Hence, the last inequality yields that

$$\lim_{n\to\infty}\|u_n\| = 0.$$

To obtain the condition, it is enough to prove that such local minima are pairwise distinct. From now on, technicality and method are different with respect to [**23**, Theorem 2.1]; see Remark 4.7 for more details.

By (k_2), there exists $\bar{\rho} > 0$ such that

$$F(\xi) > -\sigma_0\xi^p \tag{4.3}$$

for every $\xi \in]0,\bar{\rho}[$. Now, let $\{\zeta_k\} \subset]0,+\infty[$ be a sequence such that $\lim_{k\to\infty}\zeta_k = 0$ and

$$\lim_{k\to\infty}\frac{F(\zeta_k)}{\zeta_k^p} = \limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^p}. \tag{4.4}$$

Now, observe that

$$\Phi(\theta_k) \leqslant \frac{\zeta_k^p}{p}\biggl(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\biggr)\|\theta\|^p \tag{4.5}$$

for every $k \geqslant k_1$. Then, due to (4.5), one has that

$$\begin{aligned}J_\lambda(\theta_k) \leqslant{}& \frac{\zeta_k^p}{p}\biggl(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\biggr)\|\theta\|^p\\ &- \lambda\biggl(F(\zeta_k)\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x + \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)F(\theta_k(x))\,\mathrm{d}x\biggr)\end{aligned}$$

for every $k \geqslant k_1$, and, owing to (4.3), it follows that

$$\begin{aligned}J_\lambda(\theta_k) \leqslant{}& \frac{\zeta_k^p}{p}\biggl(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\biggr)\|\theta\|^p\\ &- \lambda\biggl(F(\zeta_k)\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_0\zeta_k^p\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x\biggr)\end{aligned}$$

for $k > k_2$. One then has that

$$\begin{aligned}J_\lambda(\theta_k) \leqslant \zeta_k^p\biggl[&\biggl(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\biggr)\frac{\|\theta\|^p}{p}\\ &- \lambda\biggl(\frac{F(\zeta_k)}{\zeta_k^p}\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_0\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x\biggr)\biggr]\end{aligned}$$

for every k sufficient large. But, fixing $n \in \mathbb{N}$, since

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\right)\frac{\omega_\tau^2(2^N - 1)}{\beta_0 \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x},$$

there exists a positive integer $\bar{k}$ such that $\zeta_{\bar{k}} \leqslant \xi_n'$ (thus, the function $\theta_{\bar{k}} := \zeta_{\bar{k}}\theta$ belongs to $\mathbb{E}_n$) and $J_\lambda(\theta_{\bar{k}}) < 0$. At this point, since

$$\alpha_n = J_\lambda(u_n) = \inf_{u \in \mathbb{E}_n} J_\lambda(u) \leqslant J_\lambda(\theta_{\bar{k}}) < 0,$$

bearing in mind (4.2), there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, of pairwise distinct elements. Now, clearly $\{u_n\}$ is a sequence of weak solutions for the truncated problem (K_λ^g). On the other hand, we have that

$$0 = \operatorname*{ess\,inf}_{x \in \Omega} u_n(x) < \operatorname*{ess\,sup}_{x \in \Omega} u_n(x) \leqslant \bar{t}$$

for every $n \in \mathbb{N}$. In conclusion, $\{u_n\}$ is a sequence of weak solutions for the initial problem (K_λ). □

Remark 4.2. We emphasize that, also when $1 < p \leqslant N$, no restriction on the growth of f, related to the critical exponent, is assumed in Theorem 4.1 if $a > 0$. Moreover, Theorem 1.1 follows immediately from the above result.

Remark 4.3. In analogy to Remark 3.3, (k_2) and (k_3) can be replaced by the relation

$$-\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} < \delta_{\Omega,p}^0 \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^p}, \tag{G_0}$$

where

$$\delta_{\Omega,p}^0 := \frac{\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

From (G_0) there exists $\sigma_0 > 0$ such that, for every

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(a + \frac{b}{2} + c_p^p\|\alpha\|_\infty\right)\frac{\omega_\tau(2^N - 1)}{\beta_0 \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p\,\mathrm{d}x},$$

problem (K_λ) admits a sequence $\{u_n\}$ of non-negative and non-trivial weak solutions strongly convergent to 0 in X and such that $\lim_{n\to\infty} \|u_n\|_\infty = 0$.

The following example is a direct consequence of Theorem 4.1.

Example 4.4. Fix α and σ_0 to be two positive real constants, with $\sigma_0 < \alpha$. Set

$$a_n := \frac{1}{n!n} \quad \text{and} \quad b_n := \frac{1}{n!}$$

for every $n \geqslant 2$, and define $f\colon \mathbb{R} \to \mathbb{R}$ as

$$f(t) := \begin{cases} 4\alpha(b_n^2 - b_{n+1}^2)\dfrac{t - b_{n+1}}{(a_n - b_{n+1})^2} & \text{if } b_{n+1} \leqslant t \leqslant \dfrac{a_n + b_{n+1}}{2}, \\ 4\alpha(b_n^2 - b_{n+1}^2)\dfrac{a_n - t}{(a_n - b_{n+1})^2} & \text{if } \dfrac{a_n + b_{n+1}}{2} < t \leqslant a_n, \\ 0 & \text{otherwise.} \end{cases}$$

As observed in [**21**], one has that

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = \alpha \quad \text{and} \quad \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = +\infty.$$

Moreover,

$$F(a_{n+1}) = \sup_{\xi \in [a_{n+1}, b_{n+1}]} F(\xi).$$

Then, from Theorem 4.1, for every $\lambda > 0$, the problem

$$-\biggl(a + b\int_\Omega |\nabla u|^2 \,\mathrm{d}x\biggr)\Delta u = \lambda f(u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0$$

admits a sequence $\{u_n\}$ of non-negative weak solutions strongly convergent to 0 in $H_0^1(\Omega)$ and such that $\lim_{n\to\infty} \|u_n\|_\infty = 0$.

We end this section with analogous statements to Proposition 3.9 and Example 3.10, written for the behaviour of the potential at 0.

Proposition 4.5. *Let $\{a_n\}$, $\{b_n\}$ be two sequences in $]0, +\infty[$ such that $b_{n+1} < a_n < b_n$ (for all $n \geqslant n_0$, for some $n_0 \in \mathbb{N}$), $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} b_n/a_n = +\infty$. Moreover, let $\varphi_1, \varphi_2 \in C^1([0,1])$ be two non-negative and non-zero functions such that $\varphi_i(0) = \varphi_i(1) = \varphi_i{}'(0) = \varphi_i{}'(1) = 0$ for $i = 1, 2$. Furthermore, let $g\colon \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$g(t) := \begin{cases} \varphi_1\biggl(\dfrac{t - b_{n+1}}{a_n - b_{n+1}}\biggr) & \textit{if } t \in \displaystyle\bigcup_{n \geqslant n_0} [b_{n+1}, a_n], \\ -\varphi_2\biggl(\dfrac{t - a_{n+1}}{b_{n+1} - a_{n+1}}\biggr) & \textit{if } t \in \displaystyle\bigcup_{n \geqslant n_0}]a_{n+1}, b_{n+1}[, \\ 0 & \textit{otherwise.} \end{cases}$$

Assume that there exists a constant $\sigma_0 > 0$ such that $\max_{s\in[0,1]} \varphi_2(s) < \sigma_0$ and

$$\max_{s\in[0,1]} \varphi_1(s) > \frac{\sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p \,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

Then, for every

$$\lambda > \frac{2^{p-N}}{p\tau^p}\biggl(a + \frac{b}{2}\biggr)\frac{\omega_\tau(2^N - 1)}{\max_{s\in[0,1]} \varphi_1(s) \int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - \sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p \,\mathrm{d}x},$$

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the problem

$$-\left(a+b\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u = \lambda h(x) y(u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

where

$$y(u) := |u|^{p-1}(pg(u) + ug'(u)),$$

admits a sequence $\{u_n\}$ *of non-negative and non-trivial weak solutions strongly convergent to* 0 *in* X *and such that* $\lim_{n\to\infty}\|u_n\|_\infty = 0$.

Example 4.6. Let $\Omega \subset \mathbb{R}^3$ be an open set of smooth boundary and let $h \in L^\infty(\Omega)$ such that $\operatorname{ess\,inf}_{x\in\Omega} h(x) > 0$. Furthermore, take $\{a_n\}$ and $\{b_n\}$ to be two real sequences, as in Example 4.4. Now, define $\varphi_1, \varphi_2 \in C^1([0,1])$ as follows:

$$\varphi_1(s) := \alpha \mathrm{e}^4 \mathrm{e}^{1/s(s-1)}, \quad \varphi_2(s) := \beta \mathrm{e}^4 \mathrm{e}^{1/s(s-1)} \quad \forall s \in [0,1],$$

where $\beta > 0$ and

$$\alpha > \frac{(\beta+1)\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^2 \,\mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x}.$$

Set

$$r(t) := \begin{cases} \varphi_1\left(\dfrac{t - 1/(n+1)!}{1/(n!n) - 1/(n+1)!}\right) & \text{if } t \in A, \\ -\varphi_2\left(\dfrac{t - 1/((n+1)!(n+1))}{1/(n+1)! - 1/((n+1)!(n+1))}\right) & \text{if } t \in B, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A := \bigcup_{n\geqslant 2}\left[\frac{1}{(n+1)!}, \frac{1}{n!n}\right] \quad \text{and} \quad B := \bigcup_{n\geqslant 2}\left]\frac{1}{(n+1)!(n+1)}, \frac{1}{(n+1)!}\right[.$$

Then, for every

$$\lambda > \frac{2^{-N}}{\tau^2}\left(a+\frac{b}{2}\right)\frac{\omega_\tau(2^N-1)}{\alpha\int_{B(x_0,\tau/2)} h(x)\,\mathrm{d}x - (\beta+1)\int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^2\,\mathrm{d}x},$$

the problem

$$-\left(a+b\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = \lambda h(x) y(u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

where

$$y(u) := u(2r(u) + ur'(u)),$$

admits a sequence $\{u_n\}$ of non-negative and non-trivial weak solutions strongly convergent to 0 in $H_0^1(\Omega)$ and such that $\lim_{n\to\infty}\|u_n\|_\infty = 0$.

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Remark 4.7. As pointed out in § 1, in [**23**] He and Zou studied the existence of a sequence of weak solutions for (K_λ). We note that the result in [**23**] can be easily rewritten for Kirchhoff-type problems involving the p-Laplacian operator without technical difficulties. For instance, we can prove (for every $\lambda > 0$) the existence of infinitely small weak solutions of (K_λ), requiring in Theorem 4.1 the following condition instead of (k_2) and (k_3).

(jj) There exist a constant $M \geqslant 0$ and a sequence $\{t_n\} \subset \mathbb{R}^+$, with $\lim_{n\to\infty} t_n = 0$, such that

$$\lim_{n\to\infty} \frac{F(t_n)}{t_n^p} = +\infty$$

and

$$\inf_{\xi\in[0,t_n]} F(\xi) \geqslant -MF(t_n).$$

We note that the proof of this fact can be shown by arguing exactly as in [**23**, Theorem 2.1], where the case $p = 2$ and $\alpha \equiv 0$ was analysed. In particular, the technical condition (jj) guarantees that, in the above notation, $\alpha_n < 0$ for every $n \in \mathbb{N}$. More precisely, fix $n_0 \in \mathbb{N}$, and choose a compact set $K \subset \Omega$ with $\operatorname{meas}(K) = (M+1)\operatorname{meas}(\Omega \setminus K)$ and a function $v \in X$ such that

$$v(x) := \begin{cases} 1 & x \in K, \\ 0 \leqslant v(x) \leqslant 1 & x \in \Omega \setminus K, \\ 0 & \text{otherwise.} \end{cases}$$

We now fix $\lambda > 0$. By the former condition of (jj), there exist $\bar{n} \in \mathbb{N}$ and some positive constant C such that $t_n \leqslant \xi'_{n_0}$ and

$$\operatorname*{ess\,inf}_{x\in\Omega} \int_0^{t_n} g(x,t)\,\mathrm{d}t > Ct_n^p \geqslant \frac{(M+1)}{\lambda \operatorname{meas}(K)} \Phi(t_n v)$$

for every $n \geqslant \bar{n}$, where

$$\Phi(t_n v) = \frac{t_n^p}{p}\biggl(a\int_\Omega |\nabla v(x)|^p\,\mathrm{d}x + \frac{bt_n^p}{2}\biggl(\int_\Omega |\nabla v(x)|^p\,\mathrm{d}x\biggr)^2 + \int_\Omega \alpha(x)|v(x)|^p\,\mathrm{d}x\biggr).$$

Taking into account the latter condition of (jj), for every $n \geqslant \bar{n}$, we have that

$$\begin{aligned} -\frac{\Psi(t_n v)}{\Phi(t_n v)} &= -\frac{\int_K (\int_0^{t_n} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{\Phi(t_n v)} - \frac{\int_{\Omega\setminus K}(\int_0^{t_n v(x)} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{\Phi(t_n v)} \\ &\leqslant -\frac{\int_K \operatorname{ess\,inf}_{x\in\Omega}(\int_0^{t_n} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{\Phi(t_n v)} \\ &\quad - \frac{\int_{\Omega\setminus K} \operatorname{ess\,inf}_{x\in\Omega} \inf_{t\in[0,t_n]}(\int_0^t g(x,s)\,\mathrm{d}s)\,\mathrm{d}x}{\Phi(t_n v)} \end{aligned}$$

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$$\leqslant -\frac{\int_K \text{ess inf}_{x\in\Omega}(\int_0^{t_n} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{\Phi(t_n v)} + M\frac{\int_{\Omega\setminus K} \text{ess inf}_{x\in\Omega}(\int_0^{t_n} g(x,t)\,\mathrm{d}t)\,\mathrm{d}x}{\Phi(t_n v)}$$
$$= -\frac{(1/(M+1))\,\text{meas}(K)\,\text{ess inf}_{x\in\Omega}(\int_0^{t_n} g(x,t)\,\mathrm{d}t)}{\Phi(t_n v)}$$
$$< -\frac{1}{\lambda}.$$

Whence, since $t_n v \in \mathbb{E}_{n_0}$ and $J_\lambda(t_n v) < 0$, we have that $\alpha_{n_0} := \inf_{u\in\mathbb{E}_{n_0}} J_\lambda(u) < 0$.

Our variational setting, as well as the methods used within the proof, is very similar to the one exploited in [**23**]. However, it is easy to see that Theorem 4.1 and the analogous version of [**23**, Theorem 2.1] for p-Laplacian Kirchhoff-type equations are mutually independent due to the different assumptions at 0. Indeed, requiring (jj), one immediately has that

$$\limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = +\infty.$$

We now observe that Theorem 4.1 can be applied to suitable nonlinearities f, for which the above asymptotical condition is not valid, as pointed out, for instance, in Example 4.6. On the other hand, following [**23**], one can also consider cases in which

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = -\infty.$$

Of course, this relation contradicts the assumption that

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} > -\sigma_0$$

in Theorem 4.1. For instance, adopting condition (jj) instead of (k_2) and (k_3) in Theorem 4.1, the problem

$$-\left(a + b\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)\Delta_p u = \lambda f(u) \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0,$$

where $f\colon \mathbb{R}\to\mathbb{R}$ is the function defined by

$$f(t) := \begin{cases} \left(\dfrac{p+1}{p}\right) t^{1/p}\sin t^{-1/(p+1)} - \left(\dfrac{1}{p+1}\right) t^{1/p(p+1)}\cos t^{-1/(p+1)} & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $p > (1+\sqrt{5})/2$, admits a sequence $\{u_n\}$ of non-negative and non-trivial weak solutions for every $\lambda > 0$ that is strongly convergent to 0 in X and such that $\lim_{n\to\infty}\|u_n\|_\infty = 0$. In this case, a direct computation ensures that

$$\liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^p} = -\infty,$$

where $F(\xi) := \xi^{(p+1)/p}\sin\xi^{-1/(p+1)}$, for every $\xi > 0$.

Remark 4.8. Since in our approach we just require the condition that

$$\beta_0 > \frac{\sigma_0 \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} h(x)\theta(x)^p \, \mathrm{d}x}{\int_{B(x_0,\tau/2)} h(x) \, \mathrm{d}x},$$

together with a suitable restriction on the value of the parameter λ, we emphasize that, contrary to the result contained in [**23**], we can also consider classes of problems in which

$$\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} < +\infty.$$

See, for instance, Proposition 4.5.

5. The autonomous Dirichlet problem with p-Laplacian

This last section is devoted to the study of the autonomous Dirichlet problem (D_λ^f),

$$\begin{aligned} -\Delta_p u &= \lambda f(u) \quad \text{in } \Omega, \\ u|_{\Omega} &= 0. \end{aligned}$$

First of all, conserving our previous notation, we note that the value of

$$I_p := \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} \theta(x)^p \, \mathrm{d}x$$

can be easily computed, yielding

$$I_p = \frac{2^p}{\tau^p} \int_{B(x_0,\tau)\setminus B(x_0,\tau/2)} (\tau - |x - x_0|)^p \, \mathrm{d}x = 2^p N \omega_\tau B_{(1/2,1)}(N, p+1),$$

where $B_{(1/2,1)}(N, p+1)$ denotes the *generalized incomplete beta function* given by

$$B_{(1/2,1)}(N, p+1) := \int_{1/2}^{1} t^{N-1}(1-t)^p \, \mathrm{d}t;$$

see, for instance, [**4**] for a direct computation.

The first result that we present here can be viewed as an analogue of Theorem 3.1 written for autonomous p-Laplacian equations. More precisely, the existence of infinitely many solutions for (D_λ^f), for every λ sufficiently large, is established, requiring a suitable control at ∞ of the behaviour of the function $\xi^{-p}F(\xi)$ with respect to a suitable constant depending on the geometry of Ω.

Theorem 5.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) \geqslant 0$ such that* (h_1) *holds. Furthermore, assume that there exists a real constant $\sigma_\infty > 0$ such that*

(h_2')

$$\gamma_\infty := \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} > -\sigma_\infty,$$

(h_3')

$$\beta'_\infty := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} > 2^{N+p} N B_{(1/2,1)}(N, p+1)\sigma_\infty.$$

Then, for every

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(\frac{\omega_\tau(2^N-1)}{\beta'_\infty\omega_{\tau/2} - 2^p N\omega_\tau B_{(1/2,1)}(N,p+1)\sigma_\infty}\right),$$

problem (D^f_λ) *admits an unbounded sequence* $\{u_n\}$ *of non-negative weak solutions in* X.

Remark 5.2. Note that Obersnel and Omari [**35**, Theorem 2.2] proved the existence of two sequences of solutions for the Dirichlet problem (for $p = 2$) under some constraints on the potential at ∞. One of their hypotheses implies a sign condition on f. More precisely, the nonlinearity is assumed to be definitively positive on the real half-line. Clearly, this assumption cannot be verified in our setting due to the presence of condition (h_1).

Corollary 5.3. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function with* $f(0) \geqslant 0$ *such that* (h_1) *holds. Furthermore, assume that*

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} = 0 \quad \text{and} \quad \beta'_\infty := \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} > 0.$$

There then exists $\sigma_\infty > 0$ *such that, for every*

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(\frac{\omega_\tau(2^N-1)}{\beta'_\infty\omega_{\tau/2} - 2^p N\omega_\tau B_{(1/2,1)}(N,p+1)\sigma_\infty}\right),$$

problem (D^f_λ) *admits an unbounded sequence* $\{u_n\}$ *of non-negative weak solutions in* X.

Remark 5.4. We explicitly observe that in the case $1 < p \leqslant N$, in Theorem 5.1 and Corollary 5.3, we tacitly assume that condition (h_∞) is verified.

Remark 5.5. We recall that in [**12**] and, subsequently, in [**5**], Cammaroto *et al*. and Bonanno, respectively, through a different approach and taking advantage of the compact embedding of $X \hookrightarrow C^0(\bar{\Omega})$, when $p > N$, studied the Dirichlet problem (D^f_λ). Clearly, their results cannot be applied to the case $1 < p \leqslant N$. In any case, in the low-dimensional context our theorems are also mutually independent with respect to others obtained in the cited works, since we do not assume that (1.3) holds true. Furthermore, contrary to [**5**, Theorem 1.1], in Theorem 5.1 the interval of parameters for which (D^f_λ) admits infinitely many weak solutions is always unbounded.

Remark 5.6. We emphasize that in Theorem 5.1, in order to obtain infinitely many weak solutions for λ sufficiently large, we require an oscillating behaviour of the potential F at ∞ (expressed by (h_1)) in addition to the strict algebraic inequality

$$-\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} < \delta^\infty_{N,p}\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p},$$

where $\delta^\infty_{N,p} := 2^{N+p}NB_{(1/2,1)}(N,p+1)$.

The next theorem is an immediate consequence of Theorem 4.1.

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Theorem 5.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, with $f(0) = 0$, such that* (k_1) *and* (k_2) *hold in addition to*

$$\beta_0 := \limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} > 2^{N+p} N B_{(1/2,1)}(N, p+1)\sigma_0.$$

Then, for every

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(\frac{\omega_\tau(2^N - 1)}{\beta_0 \omega_{\tau/2} - 2^p N \omega_\tau B_{(1/2,1)}(N, p+1)\sigma_0}\right),$$

problem (D_λ^f) *admits a sequence* $\{u_n\}$ *of non-negative and non-trivial weak solutions strongly convergent to* 0 *in* X *and such that* $\lim_{n\to\infty} \|u_n\|_\infty = 0$.

From Theorem 5.7 we derive the following result.

Corollary 5.8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$ and such that condition* (k_1) *holds. Furthermore, assume that*

$$\alpha_0 := \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} > 0.$$

There then exists $\sigma_0 > 0$ *such that, for every*

$$\lambda > \frac{2^{p-N}}{p\tau^p}\left(\frac{\omega_\tau(2^N - 1)}{\beta_0 \omega_{\tau/2} - 2^p N \omega_\tau B_{(1/2,1)}(N, p+1)\sigma_0}\right),$$

problem (D_λ^f) *admits a sequence* $\{u_n\}$ *of non-negative and non-trivial weak solutions strongly convergent to* 0 *in* X *and such that* $\lim_{n\to\infty} \|u_n\|_\infty = 0$.

In conclusion, we refer the reader to [**25**–**28**, **39**] for contributions related to the topics treated in this work.

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