



On the removal of the barotropic condition in helicity studies of the compressible Euler and ideal compressible magnetohydrodynamic equations

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(Received 14 November 2024; revised 18 January 2025; accepted 19 January 2025)

The helicity is a topological conserved quantity of the Euler equations which imposes significant constraints on the dynamics of vortex lines. In the compressible setting, the conservation law holds only under the assumption that the pressure is barotropic. Let us consider a volume V containing a compressible fluid with density ρ , velocity field \mathbf{u} and vorticity $\boldsymbol{\omega}$. We show that by introducing a new definition of helicity density $h_\rho = (\rho\mathbf{u}) \cdot \text{curl}(\rho\mathbf{u})$ the barotropic assumption on the pressure can be removed, although $\int_V h_\rho dV$ is no longer conserved. However, we show for the non-barotropic compressible Euler equations that the new helicity density h_ρ obeys an entropy-type relation (in the sense of hyperbolic conservation laws) whose flux \mathbf{J}_ρ contains all the pressure terms and whose source involves the potential vorticity $q = \boldsymbol{\omega} \cdot \nabla\rho$. Therefore, the rate of change of $\int_V h_\rho dV$ no longer depends on the pressure and is easier to analyse, as it depends only on the potential vorticity and kinetic energy as well as $\text{div} \mathbf{u}$. This result also carries over to the inhomogeneous incompressible Euler equations for which the potential vorticity q is a material constant. Therefore, q is bounded by its initial value $q_0 = q(\mathbf{x}, 0)$, which enables us to define an inverse resolution length scale λ_H^{-1} whose upper bound is found to be proportional to $\|q_0\|_\infty^{2/7}$. In a similar manner, we also introduce a new cross-helicity density for the ideal non-barotropic magnetohydrodynamic (MHD) equations.

Key words: topological fluid dynamics, MHD, electrohydrodynamics

1. Introduction

It was first shown by Helmholtz (1858) that for an ideal, barotropic fluid with conservative body forces, vortex lines are transported by the flow. Thomson (1868) (later Lord Kelvin) then recognised that any knots and linkages in these lines are conserved. Almost a century later, Moreau (1961) showed that the helicity \mathcal{H} is conserved, a result that was later proved independently by Moffatt (1969), who went further in recognising its relationship with the magnetic helicity invariant of Woltjer (1958*a*), and with the cross-helicity invariant \mathcal{H}_c of the ideal magnetohydrodynamic (MHD) equations (Woltjer 1958*b*).

In a fluid flow with velocity vector \mathbf{u} and vorticity vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ the helicity is defined to be a volume integral of the form $\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega} \, dV$. In this context, Moffatt (1969) showed explicitly that it is also a measure of the degree of vortex line linkages in a localised disturbance. This reference (Moffatt 1969) has been the foundation for a significant body of work on helicity applied to knots and linkages in both ideal fluids and MHD: see also Pouquet, Frisch & Léorat (1976), Berger & Field (1984), Moffatt (1978, 1985, 1990) and Moffatt & Ricca (1992). Moreover, it was shown in Enciso, Peralta-Salas & de Lizaaur (2016) that the helicity is the only integral invariant for a general volume-preserving flow. We refer to the paper by Moffatt (2014) for an overview of results and a wide range of references in the subject. The effect of helicity dynamics on the energy cascade, as well as the energy dissipation rate, has been investigated in Biferale, Musacchio & Toschi (2012), Capocci *et al.* (2023) and Linkmann (2018) (see references therein); see also Biferale & Titi (2013). In fact, tracking the helicity density has now become a standard diagnostic tool in the study of vortical structures in large scale numerical simulations of both incompressible Euler and Navier–Stokes flows: see Kerr (2018, 2023). For more general recent references in incompressible Navier–Stokes turbulence, see Iyer, Sreenivasan & Yeung (2019), Yeung & Ravikumar (2020) and Buaria, Pumir & Bodenschatz (2022).

In the setting of ideal compressible flows, a (nearly) universal feature in the study of helicity dynamics has been the assumption of barotropy of the fluid: that is, the pressure P is taken to be solely a function of the fluid density ρ , so iso-surfaces of pressure and density are parallel. As has been described in Thorpe, Volkert & Ziemianski (2003), the results of Helmholtz and Kelvin on vortex lines and structures were introduced to the meteorological and oceanographic communities by Bjerknes (1898) (in which rotational effects were included) through his circulation theorem, which is also called the Poincaré–Bjerknes circulation theorem (Poincaré 1893). The essential role played by barotropy had been pointed out earlier by Silberstein (1896). In meteorological reality, the validity of the barotropic assumption is somewhat limited. For instance, regions of barotropy in the atmosphere have a uniform temperature distribution and are distinguished by the absence of fronts, so the barotropic assumption is generally restricted to the tropics (Roulstone & Norbury 2013; Vallis 2019; Rogachevskii 2021), whereas in the midlatitudes the atmosphere is generally baroclinic. For instance, the Unified Model run by the UK Meteorological Office assumes that the pressure is a function of both density and temperature (Davies *et al.* 2005). The barotropic assumption has greater validity, however, in the study of stellar interiors or of the interstellar *medium* (Yokoi 2013). One common class of barotropic models used in astrophysics are polytropic fluids, where it is assumed that $P \propto \rho^\gamma$ for $\gamma \geq 1$.

In the non-barotropic case, a generalised helicity (which is a conserved quantity) has been introduced in Mobbs (1981): see also Gaffet (1985) and Salmon (1988). Similarly, a generalised cross-helicity can be introduced for the non-barotropic MHD equations (Yahalom 2017). As noted in Webb (2018; p. 183), these generalised helicities are non-local

quantities as they depend on a non-local variable, namely the Lagrangian time integral of the temperature. The goal of this paper is to introduce a generalised helicity which depends solely on local variables, even though it might not be fully conserved. Due to its local nature, the dynamics of this new generalised helicity is easier to interpret.

In particular, the aim of this paper is to explore the circumstances under which the barotropic condition can be removed and to study the consequences of its removal. In order to explain briefly how Moreau (1961) and Moffatt (1969) used the barotropic assumption, let us consider an ideal compressible fluid with velocity vector \mathbf{u} , density ρ , pressure P and vorticity $\boldsymbol{\omega} = \text{curl } \mathbf{u}$. In their standard form, without the inclusion of the (specific) internal energy (see § 2.2), the barotropic compressible Euler equations take the form

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P, \quad \frac{D\rho}{Dt} + \rho \text{div } \mathbf{u} = 0, \quad (1.1)$$

where $D/Dt = \partial_t + \mathbf{u} \cdot \nabla$ is the material derivative and P is solely a function of ρ . The equation for the vorticity is

$$\frac{D\boldsymbol{\omega}}{Dt} + \boldsymbol{\omega} \text{div } \mathbf{u} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \nabla \left(\rho^{-1} \right) \times \nabla P. \quad (1.2)$$

We note that the cross product vanishes as ∇P and $\nabla(\rho^{-1})$ are both parallel to $\nabla \rho$, due to the barotropic assumption. Therefore, the evolution of the helicity density can be written as an entropy-type relation (in the sense of hyperbolic conservation laws) of the form

$$\partial_t h + \text{div } \mathbf{J}_\pi = 0, \quad (1.3)$$

where $h = \mathbf{u} \cdot \boldsymbol{\omega}$, $\Pi(\rho) = \int_0^\rho \eta^{-1} P'(\eta) d\eta$ (where $P' = \frac{dP}{d\rho}$) and

$$\mathbf{J}_\pi = h\mathbf{u} + \boldsymbol{\omega} \left(\Pi - \frac{1}{2} |\mathbf{u}|^2 \right). \quad (1.4)$$

Thus, with suitable boundary conditions on the domain such as periodic boundary conditions, it follows that $\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega} dV$ is conserved. In fact, Moffatt (1969) showed that the time derivative of the helicity integrated over a moving domain $\mathcal{V}(t)$ transported by the fluid is equal to a perfect divergence. Therefore, its evolution on such a moving domain is determined only by boundary terms.

The above calculation illustrates the point that if the barotropic assumption is to be dropped then the helicity $\mathcal{H} = \int_V h dV$ is no longer conserved, because in (1.3) the term $\text{div}(\Pi \boldsymbol{\omega})$ is replaced by $\rho^{-1} \text{div}(\boldsymbol{\omega} P)$ and the additional term $\mathbf{u} \cdot (\nabla(\rho^{-1}) \times \nabla P)$ appears. Therefore a change in the definition of the helicity \mathcal{H} is required. It is hardly surprising that without the barotropic assumption, the property that vortex lines are transported with the flow will be lost. We introduce a new definition of helicity density h_ρ and helicity H as

$$h_\rho = (\rho \mathbf{u}) \cdot \text{curl}(\rho \mathbf{u}) = (\rho \mathbf{u}) \cdot (\rho \boldsymbol{\omega}) \quad \text{with } H = \int_V h_\rho dV. \quad (1.5)$$

In the context of helicity dynamics without the barotropic assumption, the challenge is to estimate the growth or decay of a topological quantity without explicitly imposing any assumptions on P . In § 2 of this paper, we will show that h_ρ obeys an entropy-type law:

$$\partial_t h_\rho + \text{div } \mathbf{J}_\rho = \sigma_\rho. \quad (1.6)$$

The pressure P appears only in \mathbf{J}_ρ and thus disappears under integration over a periodic domain. In the case of the inhomogeneous incompressible Euler equations, a potential physical interpretation of these formal results lies in using the helicity H and the constant

total energy E_0 to define an inverse length scale. In analogy with the Kolmogorov length scale, in § 2.1 we define this inverse length scale based on the average rate of change of the helicity as follows:

$$[\lambda_H]^{-1} = \left(\frac{\left\langle \left| \frac{dH}{dt} \right| \right\rangle}{\varrho_0^{1/2} E_0^{3/2}} \right)^{2/7}, \quad \varrho_0 = L^{-3} \int_V \rho dV. \quad (1.7)$$

The brackets $\langle \cdot \rangle := \frac{1}{T} \int_0^T \cdot dt$ denote a finite-time average, ϱ_0 denotes the domain averaged density (where the domain is the periodic domain $[0, L]^3$) and the energy is defined by

$$E_0 = \int_V \mathcal{E}_0 dV \quad \text{and} \quad \mathcal{E}_0 = \frac{1}{2} \rho |\mathbf{u}|^2. \quad (1.8)$$

The length scale λ_H could be interpreted as the smallest length scale on which there are significant variations of H and hence significant topological variations. The source term σ_ρ is twice the product of the potential vorticity $q = \boldsymbol{\omega} \cdot \nabla \rho$ and the energy density. It is shown in § 2.1 that this leads to the estimate

$$[\lambda_H]^{-1} \leq \left(\frac{4}{E_0 \rho_0} \right)^{1/7} \|q_0\|_\infty^{2/7}, \quad (1.9)$$

where $\|q_0\|_\infty$ is the spatial maximum norm of the (initial) potential vorticity, which is a material constant under the dynamics. We emphasise that for the inhomogeneous incompressible Euler equations the pressure solves an elliptic problem and the barotropic assumption is not relevant in this context, as the pressure also depends on velocity derivatives. As the canonical helicity is not a conserved quantity for these equations, the modified helicity H provides a new topological quantity whose dynamics is easy to interpret and its growth is bounded.

A similar calculation for the fully compressible Euler equations (including the specific internal energy) is explained in § 2.2 with the complication that σ_ρ has an extra term involving $\text{div } \mathbf{u}$ that weights the helicity density h_ρ . Finally, in § 3 we introduce a non-barotropic cross-helicity density $h_c = \rho \mathbf{u} \cdot \mathbf{B}$ for the ideal compressible MHD equations, in which \mathbf{B} is the magnetic field.

2. The evolution of h_ρ for both the inhomogeneous incompressible and the compressible 3-D Euler equations

It is known that strong solutions of the 3-D incompressible Euler equations can develop singularities in a finite time (Elgindi 2021): see also Drivas & Elgindi (2023) for a recent survey and references therein. In the compressible case, it is also known that smooth solutions blow up in finite time (Sideris 1985). Once singularities or shocks develop, solutions could be too irregular to perform the manipulations needed to obtain the results in this paper; for example, the vorticity $\boldsymbol{\omega}$ might no longer be a well-defined pointwise quantity. Thus, it should be understood that the results in the following sections are only valid for time intervals when sufficiently smooth solutions exist for both the Euler and ideal MHD equations. Sufficient regularity conditions for solutions of the incompressible Euler equations to conserve the helicity can be found in Cheskidov *et al.* (2008) and Boutros & Titi (2024) and references therein.

2.1. Results for the inhomogeneous incompressible 3-D Euler equations

We recall that the 3-D inhomogeneous incompressible Euler equations are given by

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P, \quad \operatorname{div} \mathbf{u} = 0, \quad \frac{D\rho}{Dt} = 0. \quad (2.1)$$

In this model, the density ρ is allowed to vary but \mathbf{u} remains divergence-free. The divergence-free condition implies that P must satisfy an elliptic equation that involves derivatives of the velocity:

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P \right) = -(\nabla \otimes \nabla) : (\mathbf{u} \otimes \mathbf{u}), \quad (2.2)$$

and so an imposition of barotropicity is invalid. The vorticity satisfies the following equation:

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \nabla \left(\rho^{-1} \right) \times \nabla P. \quad (2.3)$$

We note that the canonical helicity \mathcal{H} is not conserved by solutions of the inhomogeneous incompressible Euler equations, as the integral $\int_V \mathbf{u} \cdot [\nabla(\rho^{-1}) \times \nabla P] dV$ generally does not vanish. The new helicity H introduced in (1.5) is also not a conserved quantity, but we will find that its growth is bounded and its evolution depends only on local quantities. First we need to recall two basic identities.

Firstly, it is not difficult to show that ρ , \mathbf{u} and P obey the conservation law

$$\partial_t \mathcal{E}_0 + \operatorname{div} \{ (\mathcal{E}_0 + P) \mathbf{u} \} = 0, \quad (2.4)$$

where $\mathcal{E}_0 = \frac{1}{2} \rho |\mathbf{u}|^2$ is the energy density. Therefore, on a periodic domain the energy $E_0 = \int_V \mathcal{E}_0 dV$ is conserved.

Secondly, we recall Ertel's theorem (Ertel 1942) for the potential vorticity $q = \boldsymbol{\omega} \cdot \nabla \rho$. Using the evolution equation for $\boldsymbol{\omega}$ in (2.3), it can be shown that q satisfies

$$\frac{Dq}{Dt} = \boldsymbol{\omega} \cdot \nabla \left(\frac{D\rho}{Dt} \right) - \left[\nabla(\rho^{-1}) \times \nabla P \right] \cdot \nabla \rho = 0, \quad (2.5)$$

where terms of the type $(\nabla \rho) \cdot (\boldsymbol{\omega} \cdot \nabla \mathbf{u})$ cancel, in tandem with the vanishing of the pressure term. It therefore follows that q is a material constant.

The material derivatives of $\rho \mathbf{u}$ and $\rho \boldsymbol{\omega}$ are easily found, thereby giving

$$\frac{Dh_\rho}{Dt} = \rho^2 \mathbf{u} \cdot (\boldsymbol{\omega} \cdot \nabla \mathbf{u}) - \{ \rho \boldsymbol{\omega} \cdot \nabla P - \mathbf{u} \cdot (\nabla \rho \times \nabla P) \}. \quad (2.6)$$

We note that the terms involving the pressure in (2.6) form a perfect divergence:

$$\begin{aligned} \rho \boldsymbol{\omega} \cdot \nabla P - \mathbf{u} \cdot [\nabla \rho \times \nabla P] &= \nabla P \cdot [\rho \boldsymbol{\omega} + \nabla \rho \times \mathbf{u}] \\ &= \nabla P \cdot \operatorname{curl}(\rho \mathbf{u}) \\ &= \operatorname{div} \{ P \operatorname{curl}(\rho \mathbf{u}) \}. \end{aligned} \quad (2.7)$$

Equation (2.6) then becomes

$$\begin{aligned} \partial_t h_\rho + \operatorname{div} \{ h_\rho \mathbf{u} + P \operatorname{curl}(\rho \mathbf{u}) \} &= \rho^2 \boldsymbol{\omega} \cdot \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) \\ &= \boldsymbol{\omega} \cdot \nabla \left(\frac{1}{2} \rho^2 |\mathbf{u}|^2 \right) - |\mathbf{u}|^2 \boldsymbol{\omega} \cdot \nabla \left(\frac{1}{2} \rho^2 \right) \\ &= \operatorname{div} \left\{ \frac{1}{2} \boldsymbol{\omega} \rho^2 |\mathbf{u}|^2 \right\} - q \left(\rho |\mathbf{u}|^2 \right), \end{aligned} \quad (2.8)$$

which can be written as an entropy-type relation

$$\partial_t h_\rho + \operatorname{div} \mathbf{J}_\rho = \sigma_\rho, \quad (2.9)$$

with the flux vector \mathbf{J}_ρ and the scalar source term σ_ρ defined as

$$\mathbf{J}_\rho = h_\rho \mathbf{u} + P \operatorname{curl}(\rho \mathbf{u}) - \frac{1}{2} \boldsymbol{\omega} \rho^2 |\mathbf{u}|^2, \quad \sigma_\rho = -q \rho |\mathbf{u}|^2. \quad (2.10)$$

One can therefore infer that the sign of the potential vorticity impacts whether h_ρ increases or decreases. After integration of (2.9) over a periodic domain, the term $\operatorname{div} \mathbf{J}_\rho$ disappears and one finds

$$\frac{dH}{dt} = -2 \int_V q \mathcal{E}_0. \quad (2.11)$$

The property that q is a material constant means that the growth or decay of H is bounded. In fact, (2.5) implies that $\|q(\cdot, t)\|_\infty \leq \|q_0\|_\infty$. This immediately implies the bound

$$\left| \frac{dH}{dt} \right| \leq 2 \|q_0\|_\infty E_0, \quad (2.12)$$

where $\|q\|_\infty$ is the maximum spatial norm of the potential vorticity. In turn, this bound constrains the globally averaged alignment between \mathbf{u} and $\boldsymbol{\omega}$ and hence the topological dynamics.

Next we notice that the evolution of H induces a length scale λ_H of the following form:

$$[\lambda_H]^{-1} = \left(\frac{\left\langle \left| \frac{dH}{dt} \right| \right\rangle}{\varrho_0^{1/2} E_0^{3/2}} \right)^{2/7}, \quad (2.13)$$

where ϱ_0 is the average density $\varrho_0 = L^{-3} \int_V \rho \, dV$. The quantity λ_H can be interpreted as a resolution length scale: for example, the smallest length scale on which there is significant topological dynamics. We can now use (2.12) to find the following upper bound on this inverse length scale:

$$\lambda_H^{-1} \leq \left(\frac{4}{E_0 \varrho_0} \right)^{1/7} \|q_0\|_\infty^{2/7}. \quad (2.14)$$

As has been said before, λ_H could be viewed as a cutoff length scale for the helicity dynamics.

2.2. Results for the fully compressible 3-D Euler equations

The fully compressible Euler equations, including the specific internal energy e (per unit mass), require an equation for e in addition to those given in (1.1). They are given by

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P, \quad \frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0, \quad \rho \frac{De}{Dt} = -P \operatorname{div} \mathbf{u}. \quad (2.15)$$

Before considering the helicity density h_ρ , let us consider the well-known formula for the full energy density:

$$\mathcal{E} = \rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right). \quad (2.16)$$

It is not difficult to show that \mathcal{E} satisfies the exact continuity equation (2.4) (with \mathcal{E}_0 replaced by \mathcal{E}). Thus we find that the total energy $E = \int_V \mathcal{E} \, dV$ is constant for any equation

of state. For the system (2.15) to be fully determined, an equation of state for P in terms of e and ρ is required. Our results, however, are independent of the choice of equation of state and therefore we do not fix a choice. We note that similar results as described below can also be obtained for different formulations of the compressible Euler equations involving temperature or entropy dynamics. This is because our results rely only on the form of the density and velocity equations. In addition, we note that the potential vorticity q is not a material constant for these equations. However, it was shown in Gibbon & Holm (2012) that q satisfies

$$\partial_t q + \operatorname{div}(q\mathbf{u}) + \operatorname{div}[\boldsymbol{\omega}\rho\operatorname{div}\mathbf{u}] = 0. \quad (2.17)$$

One can deduce from (2.15) that $\rho\mathbf{u}$ and $\rho\boldsymbol{\omega}$ evolve according to

$$\frac{D(\rho\mathbf{u})}{Dt} + (\rho\mathbf{u})\operatorname{div}\mathbf{u} = -\nabla P, \quad (2.18)$$

$$\frac{D(\rho\boldsymbol{\omega})}{Dt} + 2(\rho\boldsymbol{\omega})\operatorname{div}\mathbf{u} = \rho\boldsymbol{\omega} \cdot \nabla\mathbf{u} + \rho^{-1}\nabla\rho \times \nabla P. \quad (2.19)$$

Following the grouping of the pressure terms as in (2.7), from (2.18) and (2.19) we deduce that

$$\partial_t h_\rho + 2h_\rho\operatorname{div}\mathbf{u} + \operatorname{div}\left\{h_\rho\mathbf{u} + P\operatorname{curl}(\rho\mathbf{u}) - \frac{1}{2}\boldsymbol{\omega}\rho^2|\mathbf{u}|^2\right\} = -q\rho|\mathbf{u}|^2. \quad (2.20)$$

With \mathbf{J}_ρ and σ_ρ defined in (2.10) and $\tilde{\sigma}_\rho = \sigma_\rho - 2h_\rho\operatorname{div}\mathbf{u}$, we find a similar entropy-type relation to (2.9):

$$\partial_t h_\rho + \operatorname{div}\mathbf{J}_\rho = \tilde{\sigma}_\rho. \quad (2.21)$$

Equation (2.21) integrates to

$$\frac{dH}{dt} + 2\int_V h_\rho\operatorname{div}\mathbf{u}\,dV = -2\int_V q\mathcal{E}_0\,dV. \quad (2.22)$$

From this equation, one can observe that the helicity H increases in regions of compression, while it decreases in regions of dilatation. Moreover, one can deduce an inequality of the following form:

$$\left|\frac{dH}{dt}\right| \leq 2\|q(\cdot, t)\|_\infty E_0 + 2\|h_\rho(\cdot, t)\|_\infty \int_V |\operatorname{div}\mathbf{u}(\cdot, t)|\,dV, \quad (2.23)$$

where we observe that bounds of this type are particularly useful in the perturbative regime of slightly compressible flows where the last term on the right-hand side could be small.

3. Cross-helicity in ideal compressible MHD

Let us consider the 3-D compressible ideal MHD equations. These are composed of

$$\rho\frac{D\mathbf{u}}{Dt} = (\operatorname{curl}\mathbf{B}) \times \mathbf{B} - \nabla P \quad \text{and} \quad \frac{D\rho}{Dt} + \rho\operatorname{div}\mathbf{u} = 0, \quad (3.1)$$

together with the induction equation for the magnetic field \mathbf{B} (as well as the divergence-free condition $\operatorname{div}\mathbf{B} = 0$) and the equation for the specific internal energy e

$$\partial_t \mathbf{B} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}) = \mathbf{B} \cdot \nabla\mathbf{u} - \mathbf{B}\operatorname{div}\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{B}, \quad \rho\frac{De}{Dt} = -P\operatorname{div}\mathbf{u}. \quad (3.2)$$

The energy density \mathcal{E}_B is an extension of (2.16) to include \mathbf{B} :

$$\mathcal{E}_B = \frac{1}{2}|\mathbf{B}|^2 + \mathcal{E}, \quad (3.3)$$

where $\mathcal{E} = \rho(\frac{1}{2}|\mathbf{u}|^2 + e)$ is the fluid energy density. We then find that

$$\partial_t \mathcal{E}_B + \operatorname{div} \left\{ \mathbf{u} (\mathcal{E}_B + P) + \frac{1}{2} |\mathbf{B}|^2 \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right\} = 0, \quad (3.4)$$

from which it follows that the full energy

$$E_{0,B} = \int_V \left[\frac{1}{2} |\mathbf{B}|^2 + \rho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \right] dV \quad (3.5)$$

is conserved.

The canonical cross-helicity $\mathcal{H}_c = \int_V \mathbf{u} \cdot \mathbf{B} dV$ is a pseudo-scalar (Pouquet *et al.* 1976; Moffatt 1978; Berger & Field 1984). In parallel with (1.5), we introduce the following generalised cross-helicity:

$$H_c = \int_V \rho \mathbf{u} \cdot \mathbf{B} dV, \quad (3.6)$$

with the cross-helicity density defined as $h_c = \rho \mathbf{u} \cdot \mathbf{B}$. The MHD equivalent of the potential vorticity is $q_c = \mathbf{B} \cdot \nabla \rho$. One can check that the magnetic potential vorticity satisfies

$$\partial_t q_c + \operatorname{div} (\mathbf{u} q_c) + \operatorname{div} (\rho \operatorname{div} \mathbf{u} \mathbf{B}) = 0. \quad (3.7)$$

The equation for $\rho \mathbf{u}$ is

$$\frac{D(\rho \mathbf{u})}{Dt} + \rho \mathbf{u} \operatorname{div} \mathbf{u} = (\operatorname{curl} \mathbf{B}) \times \mathbf{B} - \nabla P, \quad (3.8)$$

from which we deduce that

$$\begin{aligned} \partial_t h_c &= -\mathbf{B} \cdot \{ \mathbf{B} \times \operatorname{curl} \mathbf{B} + \nabla P + \rho \mathbf{u} \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla (\rho \mathbf{u}) \} \\ &\quad + \rho \mathbf{u} \cdot \{ \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \operatorname{div} \mathbf{u} \} \end{aligned} \quad (3.9)$$

$$= -2h_c \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla h_c - \mathbf{B} \cdot \nabla P + \rho \mathbf{B} \cdot \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right). \quad (3.10)$$

After rearrangement, we find

$$\partial_t h_c + h_c \operatorname{div} \mathbf{u} + \operatorname{div} \mathbf{J}_c = -\frac{1}{2} q_c |\mathbf{u}|^2. \quad (3.11)$$

The equivalent of (2.10) is

$$\mathbf{J}_c = h_c \mathbf{u} + P \mathbf{B} - \frac{1}{2} \rho \mathbf{B} |\mathbf{u}|^2, \quad (3.12)$$

so with the definition $\sigma_c = -\frac{1}{2} q_c |\mathbf{u}|^2 - h_c \operatorname{div} \mathbf{u}$ we find

$$\partial_t h_c + \operatorname{div} \mathbf{J}_c = \sigma_c. \quad (3.13)$$

This integrates to

$$\frac{dH_c}{dt} + \int_V h_c \operatorname{div} \mathbf{u} dV = -\frac{1}{2} \int_V q_c |\mathbf{u}|^2 dV. \quad (3.14)$$

As before, we note that the sign of the potential vorticity impacts the evolution of the cross-helicity H_c . Moreover, in regions of expansion H_c is decreasing, while it is increasing in regions of compression. Equation (3.13) is the equivalent of (2.21). On the left-hand side, $\operatorname{div} \mathbf{u}$ weights the cross-helicity density. In the case of the inhomogeneous incompressible MHD equations, one finds a similar relation to (2.9) and deduces that q_c is a material constant.

	\mathcal{E}	h	\mathbf{J}	σ
Baro-Euler	\mathcal{E}_0	$h = \mathbf{u} \cdot \boldsymbol{\omega}$	$\mathbf{J}_\pi = h\mathbf{u} + \boldsymbol{\omega}(\Pi - \frac{1}{2} \mathbf{u} ^2)$	0
II-Euler	\mathcal{E}_0	$h_\rho = \rho \mathbf{u} \cdot \rho \boldsymbol{\omega}$	$\mathbf{J}_\rho = h_\rho \mathbf{u} + P \text{curl}(\rho \mathbf{u}) - \frac{1}{2} \boldsymbol{\omega} \rho^2 \mathbf{u} ^2$	$\sigma_\rho = -q \rho \mathbf{u} ^2$
Comp-Euler	\mathcal{E}	$h_\rho = \rho \mathbf{u} \cdot \rho \boldsymbol{\omega}$	$\mathbf{J}_\rho = h_\rho \mathbf{u} + P \text{curl}(\rho \mathbf{u}) - \frac{1}{2} \boldsymbol{\omega} \rho^2 \mathbf{u} ^2$	$\tilde{\sigma}_\rho = \sigma_\rho - 2h_\rho \text{div} \mathbf{u}$
MHD	$\mathcal{E} + \frac{1}{2} \mathbf{B} ^2$	$h_c = \rho \mathbf{u} \cdot \mathbf{B}$	$\mathbf{J}_c = h_c \mathbf{u} + \mathbf{B}(P - \frac{1}{2}\rho \mathbf{u} ^2)$	$\sigma_c = -\frac{1}{2}q_c \mathbf{u} ^2 - h_c \text{div} \mathbf{u}$

Table 1. Summary of the entropy-type relations $\partial_t h + \text{div} \mathbf{J} = \sigma$ for the four different cases: the barotropic compressible Euler equations (Baro-Euler), the inhomogeneous incompressible Euler equations (II-Euler), the fully compressible Euler equations (Comp-Euler) and the ideal compressible MHD equations. Note that $\mathcal{E}_0 = \frac{1}{2}\rho |\mathbf{u}|^2$ and $\mathcal{E} = \rho(\frac{1}{2}|\mathbf{u}|^2 + e)$.

4. Comments and conclusion

Before making some comments, let us summarise what we have found so far. Until now, most results on helicity dynamics for compressible flows have required the barotropic approximation. The main thread in this paper has been the investigation of a different definition of helicity density for compressible flows which takes the form $h_\rho = (\rho \mathbf{u}) \cdot (\rho \boldsymbol{\omega})$ for which the barotropic condition is no longer required. The entries in table 1 show how the four cases are related. The inclusion of the ρ^2 term is crucial, as the results do not hold for other powers of ρ in the helicity density. Moreover, we repeat that the evolution of H (and H_c) can only be analysed provided there exist time intervals on which sufficiently smooth solutions exist for either the compressible Euler equations (including the internal energy), the equations of compressible ideal MHD or the inhomogeneous incompressible Euler equations, respectively.

The inclusion of the specific internal energy per unit mass e , while it does not appear explicitly in the calculations for the dynamics of h_ρ , impacts not only the \mathcal{E}_0 term through its appearance in the total energy equation (2.16), but also the source term σ_ρ defined in (2.10). For the compressible Euler equations as given in (2.15) to be fully determined, an equation of state for specifying the pressure P is required. However, our results in this paper are independent of the choice of such an equation of state, which is why the dependence of the pressure on the density, temperature and entropy has been left unspecified. A typical non-barotropic choice might be the use of the ideal gas law, where P would be a function of both density and temperature (Davies *et al.* 2005).

In the case of the inhomogeneous incompressible Euler equations, the new definition of h_ρ and its behaviour leads us to introduce a resolution length scale λ_H in (1.7). This new length scale λ_H is bounded from below in (2.14) and it suggests a typical length scale on which helicity (and hence topological) variations occur. The two main features of the dynamics are the predominance of the potential vorticity q which remains bounded in L^∞ and in the fully compressible case the (average) sign of $\text{div} \mathbf{u}$.

Acknowledgements. The authors are grateful to K. Moffatt (Cambridge), R. Pandit (IISc Bangalore), I. Roulstone (Surrey) and D. Vincenzi (Nice) for helpful and encouraging discussions on this subject. The authors thank R.M. Kerr (Warwick) for permission to use the figure with the helicity of a trefoil knot. They would also like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme ‘Anti-diffusive dynamics: from sub-cellular to astrophysical scales’ where work on this paper was undertaken.

Funding. D.W.B. acknowledges support from the Cambridge Trust and the Cantab Capital Institute for Mathematics of Information. This work was supported by EPSRC grant no. EP/R014604/1.

Declaration of interests. The authors report no conflict of interest.

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