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PROLONGATIONS OF LINEAR CONNECTIONS TO THE FRAME BUNDLE

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In this paper we construct the prolongation of a linear connection Γ on a manifold M to the bundle space $\underline{\underline{F}}M$ of its frame bundle, and show that such prolongated connection coincides with the so-called complete lift of Γ to $\underline{\underline{F}}M$.

Introduction

The purpose of the present paper is to construct the prolongation of a linear connection on a manifold M to the bundle space $\underline{\underline{F}}M$ of the frame bundle of M . To do this, we use Morimoto's general theory of prolongations to tangential fibre bundles of p^r-jets of M [6] particularized when $r = 1$, as well as some result stated in [2].

In §1, we briefly recall some results which will be used in the remaining sections. In §2, the prolongation of a connection on a principal fibre bundle P to the principal bundle J_p^1P of p^1-jets of P is constructed. In §3, we apply the results in §2 for the case of linear connections and construct the prolongation $\tilde{\Gamma}$ to $\underline{\underline{F}}M$ of a linear connection Γ on M , proving moreover that $\tilde{\Gamma}$ coincides with the so-called complete lift Γ^c of Γ defined by Mok in [5]. Finally, in §4 we show that connections adapted to G-structures on M prolongate to

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connections adapted to the corresponding prolongations of these G-structures introduced in [2].

In this paper all manifolds and mappings are assumed to be differentiable of class C^∞, entries of matrices are written as a^i_j, i being the row index and j the column index, and summation over repeated index is always implied.

1. Preliminaries

Let M be an n-dimensional manifold, R^p the Euclidean p-space and J^1_pM the set of 1-jets at $0 \in R^p$ of all differentiable mappings $f : R^p \to M$ defined on some open neighborhood of $0 \in R^p$; if $j^1(f)$ denotes the 1-jet of f at 0, the target map $\pi : J^1_pM \to M$ is defined by $\pi(j^1(f)) = f(0)$ and is in fact a projection map from J^1_pM onto M.

On J^1_pM there exists a structure of $(n+pn)$-dimensional manifold, canonically induced from the manifold structure of M, which is given as follows: let (U, x^i) be a coordinate system in M, U being the coordinate neighborhood and $\{x^i\}$ the coordinate functions on U; then, on $J^1_pU = \pi^{-1}(U)$ we define a family of coordinate functions $\{x^i, x^i_\alpha\}$ by setting

$$x^i(j^1(f)) = x^i(f(0)) , \quad x^i_\alpha(j^1(f)) = \left.\frac{\partial(x^i \circ f)}{\partial t^\alpha}\right|_0$$

$(1 \le i \le n, 1 \le \alpha \le p)$ for any $j^1(f) \in J^1_pU$, and where $(t^1, \dots, t^p)$ are the canonical coordinate functions on R^p. Then $(J^1_pU, x^i, x^i_\alpha)$ is a coordinate system in J^1_pU which will be said to be induced by (U, x^i) in M.

Let $h : M \to N$ be a differentiable map; then $h^1 : J^1_pM \to J^1_pN$ will

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denote the map canonically induced by h and given by $h^1(j^1(f)) = j^1(h \circ f)$ for any $j^1(f) \in J^1_p M$. If (U, x^i), (U', y^j) are local coordinate systems in M and N respectively, and if we assume $h : U \to U'$ expressed by $y^j = h^j(x^1, \ldots, x^n)$ then, with respect to the induced coordinate systems $\left(J^1_p U, x^i, x^i_\alpha\right)$, $\left(J^1_p U', y^j, y^j_\alpha\right)$, h^1 is expressed by

$$h^1 : y^j = h^j(x^1, \ldots, x^n) , \quad y^j_\alpha = \frac{\partial h^j}{\partial x^k} x^k_\alpha ,$$

where $1 \le k \le \dim M$, $1 \le j \le \dim N$ and $1 \le \alpha \le p$.

Let G be a Lie group; then $J^1_p G$ has also a Lie group structure, its product being defined as follows: for any $j^1(f)$, $j^1(g) \in J^1 G$, $j^1(f) \cdot j^1(g) = j^1(fg)$, where $fg : R^p \to G$ is defined by $(fg)(t) = f(t)g(t)$, $t \in \operatorname{dom} f \cap \operatorname{dom} g$. The unit element e_p of $J^1_p G$ is then the 1-jet at $0 \in R^p$ of the constant map from R^p into the unit element e of G.

Next, we shall recall some results to be used later.

(1) Assume $p = n = \dim M$. Then the bundle space $\underline{\underline{F}}M$ of the principal fibre bundle of linear frames over M (briefly, the frame bundle of M) is an open (dense) submanifold of $J^1_n M$, and the induced structure on $\underline{\underline{F}}M$ is the usual one with respect to which $\pi_M : \underline{\underline{F}}M \to M$ is a $\mathrm{Gl}(n)$-principal bundle, $\mathrm{Gl}(n)$ denoting the general linear group. If (U, x^i) is a local coordinate system in M, the induced coordinate functions on $\underline{\underline{F}}U = (\pi_M)^{-1}(U)$ will be written as $\left(x^i, X^i_j\right)$ if there is no confusion.

(2) Assume $p = 1$. Then $\pi : J^1_1 M \to M$ is nothing but the tangent bundle $\pi_M : TM \to M$. In this case, if (U, x^i) is a local coordinate system in M, the induced coordinate functions on $TU = (\pi_M)^{-1}(U)$ will be

written as $\left(x^i; \dot{x}^i\right)$. Note that the linear structure of this vector bundle is locally given as follows: let X, Y be tangent vectors at $x = \left(x^1, \ldots, x^n\right) \in U$ with coordinates $X = \left(x^i; \dot{x}^i\right)$, $Y = \left(x^i; \dot{y}^i\right)$; then $X + Y = \left(x^i; \dot{x}^i+\dot{y}^i\right)$. If $f : M \to N$, we shall denote by $Tf : TM \to TN$ the induced map.

(3) Let $P(M, \pi, G)$ be a principal fibre bundle with bundle space P , base space M , projection π and structure group G . Then $J_p^1P\left(J_p^1M, \pi^1, J_p^1G\right)$ is again a principal fibre bundle. In fact, if $\phi_U : \pi^{-1}(U) \to U \times G$ is the trivialization of P over $U \subset M$, then, since $\left(\pi^1\right)^{-1}\left(J_p^1U\right) = J_p^1\pi^{-1}(U)$, we define $\tilde{\phi}_U : J_p^1\pi^{-1}(U) \to J_p^1U \times J_p^1G$ by setting $\tilde{\phi}_U\left(j^1(f)\right) = \left(j^1(\pi \circ f), j^1\left(\eta \circ \phi_U \circ f\right)\right)$ for any $j^1(f) \in J_p^1\pi^{-1}(U)$, where $\eta : U \times G \to G$ is the canonical projection.

(4) Let $G = \mathrm{Gl}(n)$, $\left\{X_j^i\right\}$ be the canonical coordinates in $\mathrm{Gl}(n)$, $\left\{X_j^i, X_{j\alpha}^i\right\}$ the induced coordinates in $J_n^1\mathrm{Gl}(n)$ and $\left\{Y_B^A, 1 \le A, B \le n+n^2\right\}$ the canonical coordinates in $\mathrm{Gl}\left(n+n^2\right)$; then, there exists a canonical embedding of Lie groups

$$j_n : J_n^1\mathrm{Gl}(n) \to \mathrm{Gl}\left(n+n^2\right)$$

given by

$$j_n\left(\left(X_j^i, X_{j\alpha}^i\right)\right) = \begin{bmatrix} \left(X_j^i\right) & 0 & \cdots & 0 \\ \left(X_{j1}^i\right) & \left(X_j^i\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(X_{jn}^i\right) & 0 & \cdots & \left(X_j^i\right) \end{bmatrix}$$

that is, with respect to the coordinates above j_n is expressed by

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$$j_n : \begin{array}{ll} y^i_j = x^i_j \ , & y^i_{j_\alpha} = 0 \ , \\ y^{i_\alpha}_j = x^i_{j\alpha} \ , & y^{i_\alpha}_{j_\beta} = \delta^\alpha_\beta x^i_j \ , \end{array}$$

where $i_\alpha = \alpha n + i$, $1 \le i$, $\alpha \le n$. If we consider the Lie algebras of $J^1_n \mathrm{Gl}(n)$ and $\mathrm{Gl}\left(n+n^2\right)$ identified with the tangent spaces at the respective unit elements e_n and e , then the induced homomorphism

$$j_n : T_{e_n} J^1_n \mathrm{Gl}(n) \to T_e \mathrm{Gl}\left(n+n^2\right)$$

may be written as follows:

$$j_n\left(\left(\delta^i_j,\ 0;\ A^i_j,\ B^i_{j\alpha}\right)\right) = \left(\delta^A_B;\ \begin{bmatrix} \left(A^i_j\right) & 0 & ,\dots & 0 \\ \left(B^i_{j1}\right) & \left(A^i_j\right) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(B^i_{jn}\right) & 0 & \dots & \left(A^i_j\right) \end{bmatrix}\right) .$$

(5) Let $\underline{\underline{F}}M\left(M,\ \pi_M,\ \mathrm{Gl}(n)\right)$ be the frame bundle of M , $J^1_n\underline{\underline{F}}M\left(J^1_n M,\ \pi^1_M,\ J^1_n\mathrm{Gl}(n)\right)$ the induced $J^1_n\mathrm{Gl}(n)$-principal bundle and $\underline{\underline{F}}J^1_n M\left(J^1_n M,\ \pi_{J^1_n M},\ \mathrm{Gl}\left(n+n^2\right)\right)$ the frame bundle of the $\left(n+n^2\right)$-dimensional manifold $J^1_n M$. Then there exists a canonical injective homomorphism of principal bundles [2]

$$j_M : J^1_n\underline{\underline{F}}M \to \underline{\underline{F}}J^1_n M$$

over the identity of $J^1_n M$, with associate Lie group homomorphism j_n . The homomorphism j_M is locally defined as follows: let $\left(U,\ x^i\right)$ be a local coordinate system in M and consider fibered coordinate functions $\left(x^i,\ x^i_\alpha,\ X^i_j,\ X^i_{j\alpha}\right)$ on $\left(\pi^1_M\right)^{-1}\left(J^1_n U\right)$ and $\left(y^i,\ y^i_\alpha,\ Y^A_B\right)$ on $\underline{\underline{F}}J^1_n U$; then, with respect to these coordinates, j_M is expressed by

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$$j_M : \begin{array}{ll} y^i = x^i , & y^i_\alpha = x^i_\alpha , \\ y^i_j = x^i_j , & y^i_{j_\alpha} = 0 , \\ y^{i\,\alpha}_{j} = x^i_{j\alpha} , & y^{i\,\alpha}_{j_\beta} = \delta^\alpha_\beta x^i_j . \end{array}$$

Since the restriction $\underline{\underline{F}}J^1_nM\big|_{\underline{\underline{F}}M}$ of $\underline{\underline{F}}J^1_nM$ to the open submanifold $\underline{\underline{F}}M \subset J^1_nM$ is canonically isomorphic to the frame bundle $\underline{\underline{FF}}M$ of $\underline{\underline{F}}M$, then the homomorphism j_M above induces an injective homomorphism of principal bundles, noted again $j_M : J^1_n\underline{\underline{F}}M\big|_{\underline{\underline{F}}M} \to \underline{\underline{FF}}M$, over the identity of $\underline{\underline{F}}M$ and with associate Lie group homomorphism j_n.

(6) Particularizing the general results of Morimoto ([6], Chapter IV), we can assert: let M be an n-dimensional manifold; then there exist canonical diffeomorphisms

$$\alpha^{p,1}_M : TJ^1_pM \to J^1_pTM , \quad \alpha^{1,p}_M : J^1_pTM \to TJ^1_pM ,$$

such that $\alpha^{p,1}_M$ and $\alpha^{1,p}_M$ are mutually inverse. Locally, $\alpha^{p,1}_M$ is given as follows: let (U, x^i) be a local coordinate system in M and let $\left(x^i, x^i_\alpha; \dot{x}^i, \dot{x}^i_\alpha\right)$, $\left(y^i, \dot{y}^i, (y^i)_\alpha, (\dot{y}^i)_\alpha\right)$ be the induced coordinate functions on TJ^1_pU and J^1_pTU respectively. Then

$$\alpha^{p,1}_M : y^i = x^i , \quad \dot{y}^i = \dot{x}^i , \quad (y^i)_\alpha = x^i_\alpha , \quad (\dot{y}^i)_\alpha = \dot{x}^i_\alpha ,$$

with $1 \le i \le n$, $1 \le \alpha \le p$. The local expression of $\alpha^{1,p}_M$ is obvious. Moreover, if $f : M \to N$ is a differentiable map, then the following diagram is commutative

$$\begin{array}{ccc} TJ^1_pM & \underset{\alpha_M^{1,p}}{\overset{\alpha_M^{p,1}}{\rightleftarrows}} & J^1_pTM \\ {\scriptstyle Tf^1}\downarrow & & \downarrow{\scriptstyle (Tf)^1} \\ TJ^1_pN & \underset{\alpha_N^{1,p}}{\overset{\alpha_N^{p,1}}{\rightleftarrows}} & J^1_pTN \end{array}$$

2. Prolongation of connections

Let $P(M, \pi, G)$ be a principal fibre bundle and consider on P a connection whose connection form will be denoted by ω. Following Kobayashi [3], we shall consider this form ω as a differentiable map $\omega : TP \to TG$ which is a linear map of the tangent space T_uP with values in the tangent space T_eG for each point $u \in P$, and satisfying:

$$\omega(u \cdot \bar{s}) = s^{-1} \cdot \bar{s} \ ,$$

$$\omega(\bar{u} \cdot s) = s^{-1} \cdot \omega(\bar{u}) \cdot s \ ,$$

for every $u \in P$, $s \in G$, $\bar{u} \in T_uP$ and $\bar{s} \in T_sG$, and where by definition $\bar{u} \cdot s = TR_s(\bar{u})$, $u \cdot \bar{s} = TL_u(\bar{s})$, $R_s : P \to P$ and $L_u : G \to P$ being the canonical maps.

Let $\omega : TP \to TG$ be a connection form on $P(M, \pi, G)$ and define a differentiable map $\omega_1 : TJ^1_pP \to TJ^1_pG$ by setting

$$\omega_1 = \alpha_G^{1,p} \circ \omega^1 \circ \alpha_P^{p,1} \ . \tag{2.1}$$

Then, from Morimoto's general results [6], we know that

$$\mathrm{Im}\ \omega_1 \subset T_{e_p} J^1_pG \ ,$$

$$\omega_1(\tilde{u} \cdot \tilde{\bar{s}}) = \tilde{s}^{-1} \cdot \tilde{\bar{s}} \ ,$$

$$\omega_1(\tilde{\bar{u}} \cdot \tilde{s}) = \tilde{s}^{-1} \cdot \omega_1(\tilde{\bar{u}}) \cdot \tilde{s} \ ,$$

for every $\tilde{s} \in J^1_p G$, $\tilde{u} \in J^1_p P$, $\tilde{\tilde{s}} \in T_{\tilde{s}} J^1_p G$ and $\tilde{\tilde{u}} \in T_{\tilde{u}} J^1_p P$. Hence, to prove that ω_1 is actually a connection form on the principal bundle $J^1_p P\left(J^1_p M, \pi^1, J^1_p G\right)$ it suffices to prove that $\omega_1 : T_{\tilde{u}} J^1_p P \to T_{e_p} J^1_p G$ is a linear map for any $\tilde{u} \in J^1_p P$.

To do this we proceed as follows.

Let (U, x^i), (U', y^a) be local coordinate systems in P and G , respectively, with $u = \pi(\tilde{u}) \in U$, $e \in U'$ and $1 \le i \le \dim P$, $1 \le a \le \dim G$. Then, with respect to the induced coordinate systems $(TU, x^i, \dot{x}^i)$, $(TU', y^a, \dot{y}^a)$ in TP and TG respectively, ω is expressed by

$$\omega : y^a = \omega^a(x^i; \dot{x}^i) = y^a(e) , \quad \dot{y}^a = \dot{\omega}^a(x^i; \dot{x}^i) ,$$

and, therefore, for any i and a ,

$$\frac{\partial \omega^a}{\partial x^i} = \frac{\partial \omega^a}{\partial \dot{x}^i} = 0 . \tag{2.2}$$

On the other hand, if $\bar{u}, \bar{u}' \in T_u P$ are given by $\bar{u} = (x^i; \dot{x}^i)$, $\bar{u}' = (x^i; \dot{x}'^i)$ then the linearity of $\omega : T_u P \to T_e G$ implies

$$\dot{\omega}^a(x^i; \dot{x}^i + \dot{x}'^i) = \dot{\omega}^a(x^i; \dot{x}^i) + \dot{\omega}^a(x^i; \dot{x}'^i) \tag{2.3}$$

and therefore

$$\begin{aligned} \frac{\partial \dot{\omega}^a}{\partial x^i}(x^i; \dot{x}^i + \dot{x}'^i) &= \frac{\partial \dot{\omega}^a}{\partial x^i}(x^i; \dot{x}^i) + \frac{\partial \dot{\omega}^a}{\partial x^i}(x^i; \dot{x}'^i) , \\ \frac{\partial \dot{\omega}^a}{\partial \dot{x}^i}(x^i; \dot{x}^i + \dot{x}'^i) &= \frac{\partial \dot{\omega}^a}{\partial \dot{x}^i}(x^i; \dot{x}^i) + \frac{\partial \dot{\omega}^a}{\partial \dot{x}^i}(x^i; \dot{x}'^i) . \end{aligned} \tag{2.4}$$

Now, let $\left(x^i, x^i_\alpha; \dot{x}^i, \dot{x}^i_\alpha\right)$, $\left(y^a, y^a_\alpha; \dot{y}^a, \dot{y}^a_\alpha\right)$ be the induced coordinate functions on $TJ^1_p U$ and $TJ^1_p U'$ respectively. Then, taking into account the local expressions of $\alpha^{p,1}_P$, $\alpha^{1,p}_G$ and ω^1 as well as (2.2), a

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direct computation leads to the following local expression of ω_1 :

$$\omega_1 : \begin{aligned} & y^a = y^a(e) \ , \quad y^a_\alpha = 0 \ , \\ & \dot{y}^a = \dot{\omega}^a(x^i; \dot{x}^i) \ , \\ & \dot{y}^a_\alpha = \frac{\partial \dot{\omega}^a}{\partial x^k}(x^i; \dot{x}^i) \cdot x^k_\alpha + \frac{\partial \dot{\omega}^a}{\partial \dot{x}^k}(x^i; \dot{x}^i) \cdot \dot{x}^k_\alpha \ . \end{aligned}$$

Therefore, if $\tilde{u} \in J^1_p U$ has coordinates $\tilde{u} = \left(x^i, x^i_\alpha\right)$ and $\tilde{\tilde{u}}, \tilde{\tilde{u}}' \in T_{\tilde{u}} J^1_p U$ are given by $\tilde{\tilde{u}} = \left(x^i, x^i_\alpha; \dot{x}^i, \dot{x}^i_\alpha\right)$, $\tilde{\tilde{u}}' = \left(x^i, x^i_\alpha; \dot{x}'^i, \dot{x}'^i_\alpha\right)$ then $\tilde{\tilde{u}} + \tilde{\tilde{u}}' = \left(x^i, x^i_\alpha; \dot{x}^i + \dot{x}'^i, \dot{x}^i_\alpha + \dot{x}'^i_\alpha\right)$ and a straightforward computation, using (2.3) and (2.4), leads to

$$\dot{y}^a\left(\omega_1(\tilde{\tilde{u}} + \tilde{\tilde{u}}')\right) = \dot{y}^a\left(\omega_1(\tilde{\tilde{u}})\right) + \dot{y}^a\left(\omega_1(\tilde{\tilde{u}}')\right) \ ,$$

$$\dot{y}^a_\alpha\left(\omega_1(\tilde{\tilde{u}} + \tilde{\tilde{u}}')\right) = \dot{y}^a_\alpha\left(\omega_1(\tilde{\tilde{u}})\right) + \dot{y}^a_\alpha\left(\omega_1(\tilde{\tilde{u}}')\right) \ .$$

Thus we have proved the following

THEOREM 2.1. *Let* $\omega : TP \to TG$ *be a connection form on a principal fibre bundle* $P(M, \pi, G)$. *Then* $\omega_1 : TJ^1_p P \to TJ^1_p G$ *given by* (2.1) *is a connection form on the principal fibre bundle* $J^1_p P\left(J^1_p M, \pi^1, J^1_p G\right)$. *We shall call* ω_1 *the prolongation of the connection* ω *to* $J^1_p P$.

We remark that, for $p = 1$, ω_1 coincides with the connection tangential to ω due to Kobayashi ([3], p. 152), also obtained by Morimoto in [7].

3. Prolongation of linear connections to the frame bundle

In this section we apply the result in the previous section to the linear connections on a manifold. From now on the indices $h, i, j, k, \ldots, \alpha, \beta, \gamma, \ldots$ have range in $\{1, 2, \ldots, n\}$, $A, B, C, \ldots$ in $\{1, 2, \ldots, n+n^2\}$ and i_α stands for $\alpha n + i$.

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Let $\underline{\underline{F}}M(M, \pi_M, Gl(n))$ and $\underline{\underline{FF}}M\left(\underline{\underline{F}}M, \pi_{\underline{\underline{F}}M}, Gl\left(n+n^2\right)\right)$ be the frame bundles of M and $\underline{\underline{F}}M$ respectively.

THEOREM 3.1. *Let Γ be a linear connection on a manifold M. Then there exists canonically a linear connection $\tilde{\Gamma}$ on the frame bundle $\underline{\underline{F}}M$ of M, which will be called the prolongation of Γ to $\underline{\underline{F}}M$.*

Proof. Let ω be the connection form on $\underline{\underline{F}}M$ defining the connection Γ. The prolongation ω_1 of ω is a connection form on $J^1_n\underline{\underline{F}}M$, $n = \dim M$. Then, using the bundle homomorphism $j_M : J^1_n\underline{\underline{F}}M\big|_{\underline{\underline{F}}M} \to \underline{\underline{FF}}M$ described in §1, (5), we canonically obtain a connection $\tilde{\Gamma}$ on the principal fibre bundle $\underline{\underline{FF}}M$. #

Next, we shall compute the local components $\tilde{\Gamma}^A_{BC}$ of the prolongation $\tilde{\Gamma}$ of Γ to $\underline{\underline{F}}M$.

Let $\omega : T\underline{\underline{F}}M \to TGl(n)$ be the connection form of Γ, $\left(U, x^i\right)$ a local coordinate system in M, $\left(x^i, X^i_j\right)$ the induced coordinate functions on $\underline{\underline{F}}U$, $\left(y^i_j\right)$ the canonical coordinates in $Gl(n)$ and $\left(x^i, X^i_j; \dot{x}^i, \dot{X}^i_j\right)$, $\left(y^i_j; \dot{y}^i_j\right)$ the induced coordinate functions on $T\underline{\underline{F}}U$ and $TGl(n)$, respectively. Then ω is locally expressed by

$$\omega : \begin{cases} y^i_j = \omega^i_j\left(x^h, X^h_k; \dot{x}^h, \dot{X}^h_k\right) = \delta^i_j , \\ \dot{y}^i_j = \dot{\omega}^i_j\left(x^h, X^h_k; \dot{x}^h, \dot{X}^h_k\right) , \end{cases}$$

and thus, if $\left\{e^j_i\right\}$ denotes the canonical basis of $gl(n) \equiv T_eGl(n)$, we can set

$$\omega\left(x^h, X^h_k; \dot{x}^h, \dot{X}^h_k\right) = \dot{\omega}^i_j\left(x^h, X^h_k; \dot{x}^h, \dot{X}^h_k\right)e^j_i \in T_eGl(n) .$$

Let $\sigma : U \to \underline{\underline{F}}M$ be the natural cross section of $\underline{\underline{F}}M$ over U, that is $\sigma(x) = \left(x^i, \delta^i_j\right)$ for any $x = \left(x^1, \ldots, x^n\right) \in U$, and set $\omega_U = \sigma^*\omega$.

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Then ω_U defines the local components Γ^i_{jk} of Γ on U by the equation $\omega_U = \left(\Gamma^i_{jk}dx^j\right)e^k_i$ and, using Proposition 7.3 in [4], one easily finds

$$\dot{\omega}^i_j\left(x^h, X^h_k; \dot{x}^h, \dot{X}^h_k\right) = y^i_k\Gamma^k_{hl}X^l_j\dot{x}^h + y^i_h\dot{X}^h_j$$

where $\left(y^i_k\right) = \left(X^i_k\right)^{-1}$. Consequently, at the point $q = \left(x^h, X^h_k; \dot{x}^h, \dot{X}^h_k\right)$ we have

$$\frac{\partial\dot{\omega}^i_j}{\partial x^k}(q) = y^i_l\left(\partial_k\Gamma^l_{hm}\right)X^m_j\dot{x}^h ,$$

$$\frac{\partial\dot{\omega}^i_j}{\partial X^h_k}(q) = -y^i_hX^k_l\Gamma^l_{mr}X^r_j\dot{x}^m + y^i_l\Gamma^l_{mh}\dot{x}^m\delta^j_k - y^i_hX^k_m\dot{X}^m_j ,$$

$$\frac{\partial\dot{\omega}^i_j}{\partial \dot{x}^k}(q) = y^i_l\Gamma^l_{kh}X^h_j ,$$

$$\frac{\partial\dot{\omega}^i_j}{\partial \dot{X}^h_k}(q) = y^i_h\delta^j_k .$$

Now, let $\tilde{\omega}$ denote the connection form of the extension of ω_1 to $\underline{\underline{F}}J^1_nM$ via the homomorphism $j_M : J^1_n\underline{\underline{F}}M \to \underline{\underline{F}}J^1_nM$; then $j^*_M\tilde{\omega} = j_n \circ \omega_1$. If $\sigma^1 : J^1_nU \to J^1_n\underline{\underline{F}}M$ denotes the cross-section of $J^1_n\underline{\underline{F}}M$ induced by $\sigma : U \to \underline{\underline{F}}M$, then the composition $\tilde{\sigma} = j_M \circ \sigma^1$ is easily proved to be the natural cross-section of $\underline{\underline{F}}J^1_nM$ over J^1_nU.

Let $\left\{\tilde{\Gamma}^A_{BC}\right\}$ still denote the local components of the linear connection on J^1_nM which is defined by $\tilde{\omega}$, with respect to the induced coordinate system $\left(J^1_nU, x^j, x^j_\alpha\right)$. Then, if $\left\{E^A_B\right\}$ denotes the canonical basis of $\mathfrak{gl}\left(n+n^2\right) \equiv T_e\mathrm{Gl}\left(n+n^2\right)$, we have

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$$\tilde{\omega}\left(\left(\frac{\partial}{\partial x^j}\right)_{\tilde{u}}\right) = \tilde{\Gamma}^A_{jB}E^B_A \ , \quad \tilde{\omega}\left(\left(\frac{\partial}{\partial x^j_\alpha}\right)_{\tilde{u}}\right) = \tilde{\Gamma}^A_{j_\alpha B}E^B_A \ ,$$

$\tilde{u} = \tilde{\sigma}(u)$ for any point $u \in J^1_n U$. On the other hand, setting $u_1 = \sigma^1(u)$,

$$(Tj_M)\left(\left(\frac{\partial}{\partial x^j}\right)_{u_1}\right) = \left(\frac{\partial}{\partial x^j}\right)_{\tilde{u}} \ , \quad (Tj_M)\left(\left(\frac{\partial}{\partial x^j_\alpha}\right)_{u_1}\right) = \left(\frac{\partial}{\partial x^j_\alpha}\right)_{\tilde{u}} \ ,$$

and hence

$$\tilde{\omega}\left(\left(\frac{\partial}{\partial x^j}\right)_{\tilde{u}}\right) = j_n\left(\omega_1\left(\left(\frac{\partial}{\partial x^j}\right)_{u_1}\right)\right) \ ,$$

$$\tilde{\omega}\left(\left(\frac{\partial}{\partial x^j_\alpha}\right)_{\tilde{u}}\right) = j_n\left(\omega_1\left(\left(\frac{\partial}{\partial x^j_\alpha}\right)_{u_1}\right)\right) \ .$$

Then, if $u = \left(x^j, x^j_\alpha\right)$, we have

$$\begin{aligned}\omega_1\left(\left(\frac{\partial}{\partial x^j}\right)_{u_1}\right) &= \omega_1\left(x^i, I, x^i_\alpha, 0; \delta^i_j, 0, 0, 0\right) \\ &= \left(I, 0; \dot{\omega}^h_i\left(x^i, I; \delta^i_j, 0\right), \frac{\partial\dot{\omega}^h_i}{\partial x^k} x^k_\alpha\right) \\ &= \left(I, 0; \Gamma^h_{ji}, x^k_\alpha\left(\partial_k\Gamma^h_{ji}\right)\right) \ , \\ \omega_1\left(\left(\frac{\partial}{\partial x^j_\gamma}\right)_{u_1}\right) &= \omega_1\left(x^i, I, x^i_\alpha, 0; 0, 0, \delta^\alpha_\gamma\delta^i_j, 0\right) \\ &= \left(I, 0; \dot{\omega}^h_i(x^i, I; 0, 0), \frac{\partial\dot{\omega}^h_i}{\partial\dot{x}^k} \delta^k_j\delta^\alpha_\gamma\right) \\ &= \left(I, 0; 0, \delta^\alpha_\gamma\Gamma^h_{ji}\right) \ ,\end{aligned}$$

where I and 0 denote the unit matrix and the zero matrix, respectively. Therefore

$$\tilde{\omega}\left(\left(\frac{\partial}{\partial x^j}\right)_{\tilde{u}}\right) = \Gamma^h_{ji}E^i_h + x^k_\alpha\left(\partial_k\Gamma^h_{ji}\right)E^i_{h_\alpha} + \delta^\alpha_\beta\Gamma^h_{ji}E^{i_\beta}_{h_\alpha} ,$$

$$\tilde{\omega}\left(\left(\frac{\partial}{\partial x^j_\gamma}\right)_{\tilde{u}}\right) = \delta^\alpha_\gamma\Gamma^h_{ji}E^i_{h_\alpha} ,$$

and, restricting to $\underline{\underline{F}}M$, that is to the coordinate neighborhood $\underline{\underline{F}}U$, we obtain the local components of the prolongation $\tilde{\Gamma}$ of Γ to $\underline{\underline{F}}M$:

$$\tilde{\Gamma}^h_{ji} = \Gamma^h_{ji} , \quad \tilde{\Gamma}^h_{ji_\beta} = 0 , \quad \tilde{\Gamma}^h_{j_\gamma i} = 0 , \quad \tilde{\Gamma}^h_{j_\gamma i_\beta} = 0 , \quad \tilde{\Gamma}^{h_\alpha}_{j_\beta i_\gamma} = 0 ,$$

$$\tilde{\Gamma}^{h_\alpha}_{ji} = x^k_\alpha\left(\partial_k\Gamma^h_{ji}\right) , \quad \tilde{\Gamma}^{h_\alpha}_{ji_\beta} = \delta^\alpha_\beta\Gamma^h_{ji} , \quad \tilde{\Gamma}^{h_\alpha}_{j_\beta i} = \delta^\alpha_\beta\Gamma^h_{ji} .$$

Now, comparing with Mok's result in ([5], p. 81), we deduce

THEOREM 3.2. *Let* Γ *be a linear connection on* M . *Then the prolongation* $\tilde{\Gamma}$ *of* Γ *to the frame bundle* $\underline{\underline{F}}M$ *of* M *coincides with the complete lift* Γ^C *of* Γ *to* $\underline{\underline{F}}M$ *defined by* Mok [5].

4. Prolongation of connections adapted to G-structures

We begin this section proving a lemma.

LEMMA 4.1. *Let* $P(M, \pi, G)$ *be a reduced bundle of the principal fibre bundle* $P'(M, \pi, G')$, *and let* ω' *be a connection form on* P' *reducible to the connection form* ω *on* P . *Then* $J^1_pP\left(J^1_pM, \pi^1, J^1_pG\right)$ *is a reduced bundle of* $J^1_pP'\left(J^1_pM, \pi^1, J^1_pG'\right)$, *and the prolongation* ω'_1 *of* ω' *to* J^1_pP' *is reducible to the prolongation* ω_1 *of* ω *to* J^1_pP .

Proof. Let $\oint : P \to P'$ be the injective homomorphism of principal bundles which yields the reduction of G' to G , and denote also by $\oint : G \to G'$ the corresponding Lie group homomorphism. Then a straightforward computation shows that the induced bundle homomorphism $\oint^1 : J^1_pP \to J^1_pP'$ yields a reduction of J^1_pG' to J^1_pG whose associate Lie group homomorphism is the induced one, $\oint^1 : J^1_pG \to J^1_pG'$. On the other

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hand, that ω' is reducible to ω means that the following diagram commutes:

$$\begin{array}{ccc} TP & \xrightarrow{\omega} & TG \\ \downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle Tf} \\ TP' & \xrightarrow{\omega'} & TG' \end{array} .$$

Therefore, from (6) in §1 we obtain a new commutative diagram

$$\begin{array}{ccccccc} TJ^1_pP & \xrightarrow{\alpha_P^{p,1}} & J^1_pTP & \xrightarrow{\omega^1} & J^1_pTG & \xrightarrow{\alpha_G^{1,p}} & TJ^1_pG \\ \downarrow{\scriptstyle Tf^1} & & \downarrow{\scriptstyle (Tf)^1} & & \downarrow{\scriptstyle (Tf)^1} & & \downarrow{\scriptstyle Tf^1} \\ TJ^1_pP' & \xrightarrow{\alpha_{P'}^{p,1}} & J^1_pTP' & \xrightarrow{(\omega')^1} & J^1_pTG' & \xrightarrow{\alpha_{G'}^{1,p}} & TJ^1_pG' \end{array}$$

which implies that ω'_1 is reducible to ω_1 . #

Let G be a Lie subgroup of $Gl(n)$ and denote $\tilde{G} = j_n\left(J^1_nG\right) \subset Gl\left(n+n^2\right)$. Assume that $P(M, \pi, G)$ is a reduced bundle of the frame bundle $\underline{\underline{F}}M\left(M, \pi_M, Gl(n)\right)$ of M , $n = \dim M$, that is P is a G-structure on M . In [2] we have defined the prolongation of the G-structure P on M to a $\tilde{G}$-structure $\tilde{P}$ on $\underline{\underline{F}}M$ as follows: we consider the injective bundle homomorphism $i^1 : J^1_nP \to J^1_n\underline{\underline{F}}M$ induced by $i : P \to \underline{\underline{F}}M$ and define $\tilde{P} = \left(j_M \circ i^1\right)\left(J^1_nP\right)\Big|_{\underline{\underline{F}}M}$.

As usually, we say that a linear connection Γ on M is adapted to the G-structure $P(M, \pi, G)$ on M if Γ is reducible to a connection on P . Then, taking into account Lemma 4.1 and the results in the previous section, we easily deduce

THEOREM 4.2. *Let Γ be a linear connection on M adapted to a G-structure $P(M, \pi, G)$ on M . Then the prolongation $\tilde{\Gamma}$ of Γ to $\underline{\underline{F}}M$ is adapted to the $\tilde{G}$-structure $\tilde{P}(\underline{\underline{F}}M, \pi, \tilde{G})$ on FM , prolongation of P to $\underline{\underline{F}}M$.*

We remark that Theorem 4.2 improves some particular results in [1] and

[5] where only the prolongations (or complete lift) of G-structures on M defined by tensor fields of types $(0, s)$ and $(1, s)$ have been considered.

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