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RESEARCH ARTICLE

Girth Alternative for subgroups of $PL_o(I)$

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Abstract
We prove the *Girth Alternative* for finitely generated subgroups of $PL_o(I)$. We also prove that a finitely generated subgroup of $Homeo_+(I)$ which is sufficiently rich with hyperbolic-like elements has infinite girth.

1. Introduction

The notion of a girth for a finitely generated group was first introduced in [14] motivated by the study of Heegaard splittings of closed 3-manifolds.

Definition 1.1. *Let Γ be a finitely generated group. For any finite generating set S of Γ, $girth(\Gamma, S)$ will denote the minimal length of relations among the elements of S. Then, we define*

$$girth(\Gamma) = \sup_{\langle S\rangle=\Gamma, |S|<\infty} girth(\Gamma, S).$$

By definition above, an infinite cyclic group has infinite girth, but this fact should be viewed as a degeneracy since (as remarked in [1]) any group satisfying a law and non-isomorphic to $\mathbb{Z}$ has a finite girth.

In [2], we have proved that if a finitely generated group is word hyperbolic, or one-relator, or linear then it is either virtually solvable or has infinite girth. More generally, given a class $\mathcal{C}$ of finitely generated groups, we will say that $\mathcal{C}$ satisfies the *Girth Alternative* if any group from the class $\mathcal{C}$ is either virtually solvable or has infinite girth.

In [15], S. Yamagata has proved the *Girth Alternative* for convergence groups and for irreducible subgroups of mapping class groups. Independently, in [12] and [13], K. Nakamura proves the alternative for convergence groups but also for all subgroups of mapping class groups as well as for subgroups of $Out(\mathbb{F}_n)$ that contain irreducible elements with irreducible powers.

In this paper, we will prove that the *Girth Alternative* holds for subgroups of $PL_o(I)$—the group of orientation-preserving piecewise linear homeomorphisms of the closed interval $I = [0, 1]$. It is known that any virtually solvable subgroup of $PL_o(I)$ is indeed solvable (see [7], Corollary 1.3.), so the *Girth Alternative* in this case is equivalent to the following:

Theorem 1.2. *Any finitely generated subgroup of $PL_o(I)$ is either solvable or has infinite girth.*

It is remarked in [1] that a finitely generated noncyclic group which satisfies a law has a finite girth. Thus, we obtain the following corollary.

Corollary 1.3. *If a subgroup of $PL_o(I)$ satisfies a law then it is solvable.*

Remark 1.4. *As another corollary of Theorem 1.2, we obtain that girth*$(F) = \infty$ *where F denotes the R.Thompson's group. This fact has been proved in [9] and in [4]; both proofs use different ideas from each other and from the proof of Theorem 1.2. Theorem 1.2 also implies that girth*$(B) = \infty$ *where B is the Brin group introduced in [10] under the notation* G_1 *(the notation B is used in [7] and in [11], among other places).*

It is easy to prove the *Girth Alternative* for $Diff_{\omega}(I)$—the group of orientation-preserving analytic diffeomorphisms of I, however, we do not know if the alternative still holds when the regularity is decreased. The following questions are interesting to us:

Question 1. *Does* Girth Alternative *hold for subgroups*
(a) of $\mathrm{Homeo}_+(I)$*? (b) of* $\mathrm{Diff}_+(I)$*?*

Question 2. [1] *Is there a finitely generated subgroup of* $PL_o(I)$ *which is not embeddable into* $\mathrm{Diff}_+(I)$ *?*

In regard to Question 1, we prove the following partial result.

Theorem 1.5. *Let* Γ *be any finitely generated subgroup of* $\mathrm{Homeo}_+(I)$*. Assume that for all* $N \in \mathbb{N}$*, for every sequence* $0 < x_1 < x_2 < \cdots < x_N < 1$*, and for all* $\epsilon > 0$*, one can find an element* $\gamma \in \Gamma$ *such that* $Fix(\gamma) = \{0, c_1, \ldots, c_N, 1\}$*, and* $|c_i - x_i| < \epsilon$*, for all* $1 \leq i \leq N$*. Then girth*$(\Gamma) = \infty$*.*

Remark 1.6. *As a corollary of Theorem 1.5, we obtain yet another proof of the fact that girth*$(F) = \infty$*.*

At the end of this section, let us also note that the core of the ideas in our proofs (more visibly in the proof of Theorem 1.5) aligns with (and can be viewed as a case of) the method of "fast generating sets" which has been developed systematically in [5]. In [6], the authors exploit this method to build elementary amenable subgroups of $PL_o(I)$ with very complex EA class.

2. Towers and exemplary towers: review of Collin Bleak's results

In the proof of Theorem 1.2, as a crucial tool, we use the result of C. Bleak on the existence of arbitrarily high towers in a non-solvable subgroup of $PL_o(I)$, [7]. First, we would like to introduce the following notions essentially borrowed from [7], with a slightly different terminology.

Definition 2.1. *An ordered n-tuple* $(f_1, \ldots, f_n)$ *of elements of* $PL_o(I)$ *is said to form a* strict tower *if there exist intervals* (a_i, b_i), $1 \leq i \leq n$ *such that*

(i) $0 < a_1 < \ldots < a_n < b_n < \ldots < b_1 < 1$;
(ii) for all $i \in \{1, \ldots, n\}$, $f_i(a_i) = a_i, f(b_i) = b_i$*, and* f_i *has no fixed points in* (a_i, b_i).
(iii) for all $i, j \in \{1, \ldots, n\}$*, if* $i < j$ *then* $f_i(x) > f_j(x), \forall x \in [a_j, b_j]$

We will denote the strict tower by $T = [(f_1, \ldots, f_n); (a_1, b_1), \ldots, (a_n, b_n)]$*; n will be called* the height of the tower T*, and the interval* (a_i, b_i) *will be called* the i-th base *of the tower.*

Definition 2.2. *We will say that a strict tower*

$$T = [(f_1, \ldots, f_n); (a_1, b_1), \ldots, (a_n, b_n)]$$

is suitable *if for any nonzero integer p and for all* $1 \leq i < j \leq n$*, the following condition holds:* $f_i^p([a_j, b_j]) \cap \bigcup_{i+1 \leq k \leq n} supp(f_k) = \emptyset$ (1).

[1] This question has now been answered positively in [8].

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Remark 2.3. *Condition (1) of Definition 2.2 implies that for any nonzero integer p and for all* $1 \leq i < j \leq n$, $f_i^p([a_j, b_j]) \cap [a_j, b_j] = \emptyset$ (2). *Notice that, if the n-tuple* $(f_1, \ldots, f_n)$ *of elements of* $PL_o(I)$ *forms a tower then for sufficiently big* $q \in \mathbb{N}$, *the n-tuple* $(f_1^q, \ldots, f_n^q)$ *forms a tower with the same bases which satisfies condition* (2). *Also, the existence of a suitable tower of arbitrary height in non-solvable subgroups of* $PL_o(I)$ *immediately follows from the existence of the exemplary towers of arbitrary height, [7].*

To explain the existence of exemplary towers, we would like to make a digression into some of the results of C. Bleak. For the rest of this section, let $\Gamma \leq PL_o(I)$. The following notions and notations are all borrowed directly from [7].

We will call the convex hull of a point in I under the action of Γ an *orbital* of Γ, if this convex hull contains more than one point. We note that the orbitals are open intervals. If $g \in \Gamma$, we will refer to an orbital of the group $\langle g \rangle$ as an orbital of g. If an open interval A is an orbital of g, then the pair (A, g) will be called a *signed orbital of* G. g will be called the *signature* of the signed orbital (A, g).

Given a set Y of signed orbitals of G, the symbol S_Y will refer to the set of signatures of the signed orbitals in Y. Similarly, the symbol O_Y will refer to the set of orbitals of the signed orbitals of Y. We note that the set of signed orbitals of $PL_o(I)$ is a partially ordered set under the lexicographic order induced from the partial order on subsets of I (induced by inclusion) in the first coordinate, and the left total order of the elements of $PL_o(I)$ in the second coordinate.

A *tower* T of G is a set of signed orbitals which satisfies the following two criteria.

1. T is a chain in the partial order on the signed orbitals of G.
2. For any $A \in O_T$, T has exactly one element of the form (A, g).

Given a tower T of G, if $(A, g), (B, h) \in T$ then one of $A \subseteq B$ and $B \subseteq A$ holds, with equality occurring only if $g = h$ as well. Therefore, one can visualize the tower as a stack of nested levels that are always getting wider as one goes "up" the stack.

The cardinality of the set O_T will be called the *height* of the tower T. Besides the cardinality, we also want to make use of the order structure of towers which allows to define the following more sensitive notions: if there is an order-preserving injection from $\mathbb{N}$ to T, then we will say T is *tall*, and if there is an order-preserving injection from $-\mathbb{N}$ to T, then we will say T is *deep*. If T is both deep and tall, then we will say T is a *bi-infinite tower*; in the latter case, there will be an order-preserving injection from $\mathbb{Z}$ to T.

A major result of [7] is the following beautiful geometric characterization of solvable subgroups of $PL_o(I)$.

Theorem 2.4. *If* $G \leq PL_o(I)$ *is a non-solvable subgroup if and only if G admits a tower of height n for any* $n \geq 1$.

If G admits two signed orbitals (A, g) and (B, h) so that $A = (a_1, a_2)$ and $B = (b_1, b_2)$, with $a_1 < b_1 < a_2 < b_2$ then we will say that G admits a *transition chain of length two*. One can similarly define transition chains of any (finite) length, but we will have no need for that generality here.

If $A = (a, b)$ is an orbital of the group G, and G has an element g which has an orbital $B = (c, d)$ so that either $c = a$ or $d = b$, then we say that g has an orbital that shares an end with A.

Given an orbital A of H, we say that *h realizes an end of* A if some orbital of h lies entirely in A and shares an end with A. If g and h are elements of $PL_o(I)$ and there is an interval $B = (a, b) \subset I$ so that both g and h have B as an orbital, then we will say that g and h *share the orbital* B.

We will say that an orbital A of a group $H \leq PL_o(I)$ is *imbalanced* if some element $h \in H$ realizes one end of A, but not the other, and we will say A is *balanced* if whenever an element $h \in H$ realizes one end of A, then h also realizes the other end of A (note that h might do this with two distinct orbitals). A subgroup $H \leq PL_o(I)$ will be called balanced if given any subgroup $G \leq H$, and any orbital A of G,

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every element of G which realizes one end of A also realizes the other end of A. In the case where H has a subgroup G with an imbalanced orbital, then we will say that H is imbalanced.

We say a tower T is an *exemplary tower* if the following two additional properties hold:

1. Whenever $(A, g), (B, h) \in T$ then $(A, g) \le (B, h)$ implies the orbitals of g are disjoint from both ends of the orbital B.
2. Whenever $(A, g), (B, h) \in T$ then $(A, g) \le (B, h)$ implies no orbital of g in B shares an end with B.

C. Bleak proves several technical results which indicate the plethora of exemplary towers in $PL_o(I)$. The following two lemmas are stated in [7] as Lemmas 1.4 and 2.12, respectively.

Lemma 2.5. *If H is a subgroup of $PL_o(I)$, and H admits a transition chain of length two, then H admits infinite towers.*

Lemma 2.6. *If H is a subgroup of $PL_o(I)$ which does not admit a transition chain of length two and H has a tower T, then T is exemplary.*

Thus, the existence of a transition chain of length two implies the existence of infinite towers, and the absence of the transition chain of length two implies that all towers are nice, that is, exemplary. It turns out the absence of a transition chain of length two also implies certain nice properties of the group itself. The following result is stated as Remark 4.9 in [7].

Lemma 2.7. *If G is a subgroup of $PL_o(I)$ that does not admit transition chains of length two, then G is balanced.*

We also need the following two technical results from [7].

Lemma 2.8 (Corollary 2.13, [7]). *If G is a balanced subgroup of $PL_o(I)$ and G admits a tall tower in some orbital A, or G admits a deep tower in some orbital A, or G admits a bi-infinite tower in some orbital A, then G admits an exemplary tall tower in A, or G admits an exemplary deep tower in A, or G admits an exemplary bi-infinite tower in A, respectively.*

Lemma 2.9 (Lemma 2.8, [7]). *Suppose $H \le PL_o(I)$, and that $G \le H$ has an imbalanced orbital $A = (a, b)$. Then H admits an exemplary bi-infinite tower E whose orbitals are all in A.*

Corollary 2.10. *If H is an imbalanced subgroup of $PL_o(I)$, then H admits an exemplary bi-infinite tower.*

Combining the above results we can now claim the following lemma.

Lemma 2.11. *If $\Gamma \le PL_o(I)$ is not solvable, then it contains an exemplary tower of any height $n \ge 1$.*

Proof. If Γ does not admit a transition chain of length two, then the claim follows from Theorem 2.4 and Lemma 2.6. If Γ is not balanced then the claim follows from Corollary 2.10. Thus, we can assume that Γ is balanced and admits a transition chain of length two. Then by Lemma 2.5, Γ admits an infinite tower. Then by Lemma 2.8, it admits an exemplary bi-infinite tower. □

3. Proof of Theorem 1.2

For any natural number r, let $G_r = (\ldots((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \ldots \mathbb{Z}) \wr \mathbb{Z}$ where the iterated wreath product is taken r times. The group G_r can be defined inductively as $G_0 = 1, G_{i+1} = G_i \wr \mathbb{Z} := \mathbb{Z} \ltimes \bigoplus_{n\in\mathbb{Z}} G_i, 0 \le i \le r-1$.

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In the wreath product $G_i \wr \mathbb{Z}$, the standard generator of the acting group $\mathbb{Z}$ will be denoted by g_{r-i}. (in [7], the group G_r is denoted by W_r).

The following lemma will be useful; the idea of its proof is essentially borrowed from the proof of Lemma 2.3 in [1].

Lemma 3.1. *For all $q, k \in \mathbb{N}$, there exist $r \in \mathbb{N}$ and $w_1, \ldots, w_k \in G_r$ such that there is no relation of length less than q among $w_1, \ldots, w_k$.*

Proof. Since a free group on two generators contains a free group on k generators for any $k \geq 3$, it is sufficient to prove the claim for $k = 2$. We will do this by induction; more precisely, it suffices to prove the following claim: *if A, B are nontrivial groups and A satisfies no law in two variables of length less than $n \geq 4$, then the wreath product $B \wr A = A \ltimes \bigoplus_{i \in A} B$ satisfies no law in two variables of length less than $n + 1$.*

Indeed, let $w_1, w_2 \in A$ with no relation of length less than n. An element of $\bigoplus_{i \in A} B$ can be written as $(x_g)_{g \in A}$ where all but finitely many "coordinates" are 1. Let b be a non-identity element of B, and $t = (x_g)_{g \in A}$ be the element of $\bigoplus_{i \in A} B$ where $x_g = b$ for $g = 1$ and $x_g = 1$ otherwise.

Then there is no relation of length less than $n + 1$ among the elements tw_1 and w_2. Indeed, let $W(x, y)$ be a nontrivial reduced word of length $k < n + 1$ in the alphabet $\{x^{\pm 1}, y^{\pm 1}\}$ such that $W(tw_1, w_2) = 1 \in B \wr A$. For every $1 \leq i \leq k$, let also W_i be the prefix of W of length i. Then $W(w_1, w_2) = W_k(w_1, w_2) = 1 \in A$, moreover, for at least one $i \in \{1, \ldots, k\}$ there exists $j \in \{1, \ldots, k\} \backslash \{i\}$ such that $W_i(w_1, w_2) = W_j(w_1, w_2)$. Indeed, we have

$$W(tw_1, w_2) = \left(\prod_{i=1}^{s} (W_{n_i} t^{\epsilon_i} W_{n_i}^{-1}) \right) W(w_1, w_2)$$

where $1 \leq s \leq k$, $\epsilon_i \in \{-1, 1\}$, $1 \leq i \leq s$ and $0 \leq n_1 < n_2 < \ldots < n_s \leq k$ (here, we define $W_0 = 1$).

Now, we have $W_{n_i} t^{\epsilon_i} W_{n_i}^{-1} \in \bigoplus_{\alpha \in A} B$ for all $1 \leq i \leq s$. Moreover, by the re-indexing action of A on $\bigoplus_{\alpha \in A} B$, for each of the terms $W_{n_i} t^{\epsilon_i} W_{n_i}^{-1}$ we have only one nontrivial (i.e. non-identity) coordinate consisting of b or b^{-1} depending if $\epsilon_i = 1$ or $\epsilon_i = -1$ respectively. Hence, to have a necessary cancelation, we need to have two distinct $p, q \in \{1, \ldots, s\}$ such that $W_{n_p} = W_{n_q}$ (and $\epsilon_p = -\epsilon_q$). Thus, we can take $i = n_p, j = n_q$.

Thus, we established that for at least one $i \in \{1, \ldots, k\}$ there exists $j \in \{1, \ldots, k\} \backslash \{i\}$ such that $W_i(w_1, w_2) = W_j(w_1, w_2)$. But this implies a relation of length less than n among w_1 and w_2. □

Let us note that in the proof of Lemma 3.1, we are using a careful re-examination of the word $W(tw_1, w_2)$. We view the latter as a word in the alphabet $\{t^{\pm 1}, w_1^{\pm 1}, w_2^{\pm 1}\}$. There will be no consecutive occurrences of $t^{\pm 1}$; as we read through from left to right, we write $W(tw_1, w_2)$ as a product of conjugates of $t^{\pm 1}$. The residue at the very right will be the word $W(w_1, w_2)$. For example, if $W(x, y) = [x, y] = xyx^{-1}y^{-1}$, then we will have

$$\begin{aligned} W(tw_1, w_2) &= [tw_1, w_2] = tw_1 w_2 w_1^{-1} t^{-1} w_2^{-1} \\ &= (1t1)(w_1 w_2 w_1^{-1} t^{-1} w_1 w_2^{-1} w_1^{-1})[w_1, w_2]. \end{aligned}$$

We also would like to observe the following simple lemma.

Lemma 3.2. *Let $T = [(f_1, \ldots, f_n, f_{n+1}); (a_1, b_1), \ldots, (a_n, b_n), (a_{n+1}, b_{n+1})]$ be a suitable tower, and $\phi_1, \ldots, \phi_n \in PL_o(I)$ be such that for all $i \in \{1, \ldots, n\}$, $supp(\phi_i) \subseteq [a_i, b_i]$ and $\phi_i|_{[a_i, b_i]} = f_i|_{[a_i, b_i]}$. Then*

(a) the maps $\phi_1, \ldots, \phi_n$ generate a subgroup $H_n \leq PL_o(I)$ such that there exists an isomorphism $\Phi: H_n \to G_n$ given by $\Phi(\phi_i) = g_i$, $1 \leq i \leq n$;

(b) if $W = W(g_1, \ldots, g_n)$ is any non-identity element of G_n, then there exists a word $U = U(g_1, \ldots, g_n) \in G_n$ such that $W_1((U_1(a_{n+1}, b_{n+1}))) \cap U_1((a_{n+1}, b_{n+1})) = \emptyset$ where $W_1 = W(\phi_1, \ldots, \phi_n)$, $U_1 = U(\phi_1, \ldots, \phi_n)$;

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(c) if $W = W(g_1, \dots, g_n) \in G_n$ and $x \in (a_{n+1}, b_{n+1})$, then $W(f_1, \dots, f_n)(x) = W(\phi_1, \dots, \phi_n)(x)$.

Now, let Γ be a finitely generated non-solvable subgroup of $PL_o(I)$, $s = d(\Gamma)$ (i.e. s is the minimal cardinality for a generating subset of Γ). Without loss of generality, we may assume that Γ is irreducible, that is, it has no global fixed point in the interval $(0, 1)$.

For every finite generating set X of Γ, we will fix the left-invariant Cayley metric on Γ with respect to X, and let $B_k(1; X)$ denote the ball of radius k centered at identity element $1 \in \Gamma$, for all $k \geq 1$. We also let

$$S_k(\Gamma, X, c) = \{\gamma \in B_k(1; X) \mid \gamma'(0) = c\}, \; S_k(\Gamma, X) = \{\gamma \in B_k(1; X) \mid \gamma'(0) = 1\},$$

$$C_k(\Gamma, X) = \{c \in \mathbb{R}_+ \mid S_k(\Gamma, X, c) \neq \emptyset\}.$$

Fix a positive integer m, and let $q = 2m^2$. By Lemma 3.1, there exists $r \in \mathbb{N}$ such that in the group G_r there exist s elements $w_1, w_2, \dots, w_s$ such that there is no relation of length less than q among $w_1, \dots, w_s$. Let $g_1, \dots, g_r$ be the standard generators of G_r and let $w_i = W_i(g_1, \dots, g_r)$, $1 \leq i \leq s$, where W_i is a reduced word in the free group of rank r formally generated by the letters $g_1, \dots, g_r$.

Since Γ is non-solvable, the commutator subgroup $[\Gamma, \Gamma]$ is non-solvable. Then by Remark 2.3 and Lemma 2.11, there exists an ordered $(r+1)$-tuple $(f_1, \dots, f_r, f_{r+1})$ of elements of $[\Gamma, \Gamma]$ which form a suitable tower of height $r+1$. Let $0 < d < D < 1$ such that $\bigcup\limits_{1 \leq i \leq r+1} supp(f_i) \subset (d, D)$.

Then, we can find $\epsilon_0 > 0$, a finite generating set S of Γ of cardinality s, and a suitable tower $(h_1, \dots, h_r, h_{r+1})$ of elements of Γ with bases $(a_i, b_i) \subset (0, \epsilon_0)$, $1 \leq i \leq r+1$ and $\Omega = \bigcup\limits_{1 \leq i \leq r+1} supp(h_i)$ such that $S \cap S^{-1} = \emptyset$ and the following conditions hold:

- (i) for all $\beta \in B_{2m}(1; S)$, $\beta(\Omega) \subset (0, \epsilon_0)$;
- (ii) for any two distinct $c_1, c_2 \in \mathbb{R}_+$, and for all $\beta_1 \in S_{2m}(\Gamma, S, c_1)$, $\beta_2 \in S_{2m}(\Gamma, S, c_2)$, $\beta_1(\Omega) \cap \beta_2(\Omega) = \emptyset$;
- (iii) for all $c \in \mathbb{R}_+$, $\beta_1, \beta_2 \in S_{2m}(\Gamma, S, c)$, and for all $x \in \Omega$, $\beta_1(x) = \beta_2(x)$ (so, in particular, for all $\beta \in S_{2m}(\Gamma, S)$ and for all $x \in \Omega$, $\beta(x) = x$).

Indeed, let $X_0 = \{\alpha_1, \dots, \alpha_s\}$ be any generating set of Γ of cardinality s. Without loss of generality, we may assume that

$$1 \leq \alpha_1'(0) \leq \cdots \leq \alpha_s'(0) \text{ and } \alpha_s'(0) > 1.$$

Let $\delta > 0$ such that α_s has no singularity in $(0, \delta)$. By irreducibility, there exists $\phi \in \Gamma$ such that $\phi(D) < \delta$. For $n \geq 1$ and $1 \leq i \leq r+1$, we let $f_i^{(n)} = \psi_n^{-1} f_i \psi_n$, where $\psi_n = \phi^{-1}\alpha_s^n$. Let also $(d_n, D_n) = \psi_n^{-1}((d, D))$, $n \geq 1$. Notice that the interval (d_n, D_n) (in particular, its subset $\cup_{1 \leq i \leq r+1} supp(f_i^{(n)})$) converges to zero as $n \to \infty$; moreover, there exists a positive integer p such that for all $k \geq p$, we have $\frac{D_k}{d_k} = \frac{D_p}{d_p}$.

We can choose a finite generating set $S = \{\gamma_1, \dots, \gamma_s\}$ such that the inequality $\min\{c \in C_{2m}(\Gamma, S) \mid c > 1\} > \frac{D_p}{d_p}$ holds. To see this, first, we let $X_1 = \{\beta_1, \dots, \beta_s\}$ where $\beta_s = \alpha_s$, and $\beta_i = \alpha_i \alpha_s^{n_i}$, $1 \leq i \leq s-1$ for some $n_{s-1} \leq \cdots \leq n_2 \leq n_1$ such that $1 < \beta_s'(0) \leq \beta_{s-1}'(0) \leq \cdots \leq \beta_1'(0)$ and $\beta_1'(0) > \frac{D_p}{d_p}$. After this, we modify the generating set X_1 further by letting $S = \{\beta_1, \beta_1^{\lambda}\beta_2, \beta_1^{\lambda^2}\beta_3, \dots, \beta_1^{\lambda^{s-1}}\beta_s\}$ where $\lambda = 4m+1$.

Then, for sufficiently big n, we can take $h_i = f_i^{(n)}$, $1 \leq i \leq r+1$ to satisfy the claims (i)–(iii).

Now, let A_m be a minimal subset of $B_m(1;S)$ such that $A_m \cap S_m(\Gamma, S, c) \neq \emptyset$ for every $c \in C_m(\Gamma, S)$; and for all $k \in \{1, \dots, s\}$, let

$$\omega_k = \prod_{\gamma \in A_m} (\gamma v_k \gamma^{-1}), \; \eta_k = \prod_{\gamma \in A_m} (\gamma u_k \gamma^{-1})$$

where $v_k = W_k(h_1, \dots, h_r)$, $u_k = v_k^m$ and for the products (the formulas for ω_k and η_k) we choose any linear order on the set A_m.

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Notice that because of conditions (i)–(iii), for any two $\gamma', \gamma'' \in B_m(1)$, we have $[\gamma' v_k(\gamma')^{-1}, \gamma'' v_k(\gamma'')^{-1}] = 1$ and $[\gamma' u_k(\gamma')^{-1}, \gamma'' u_k(\gamma'')^{-1}] = 1$, so the order in the products $\prod_{\gamma \in A_m} (\gamma v_k \gamma^{-1})$ and $\prod_{\gamma \in A_m} (\gamma u_k \gamma^{-1})$ does not matter.

Now, let $S^{(m)} = \{\eta_1\gamma_1\eta_2, \eta_2\gamma_2\eta_3, \ldots, \eta_s\gamma_s\eta_1, \omega_1, \ldots, \omega_s\}$. Then, since $\omega_k^m = \eta_k$ for all $1 \leq k \leq s$, the set $S^{(m)}$ generates Γ, and there is no relation of length less than m among the elements of $S^{(m)}$.

Indeed, let $R = R(\eta_1\gamma_1\eta_2, \eta_2\gamma_2\eta_3, \ldots, \eta_s\gamma_s\eta_1, \omega_1, \ldots, \omega_s)$ denote such a relation. Then, we can write

$$R(\eta_1\gamma_1\eta_2, \eta_2\gamma_2\eta_3, \ldots, \eta_s\gamma_s\eta_1, \omega_1, \ldots, \omega_s) = R_0(\theta_1 R_1 \theta_1^{-1})(\theta_2 R_2 \theta_2^{-1}) \ldots (\theta_n R_n \theta_n^{-1}) R_{n+1}$$

where $n \leq m$, $\theta_i \in B_m(1)$ for all $1 \leq i \leq n$, and $R_j = R_j(\eta_1, \ldots, \eta_s, \omega_1, \ldots, \omega_s)$ is a reduced word of length at most m for all $0 \leq j \leq n+1$; moreover, $R_1, \ldots, R_n$ are nontrivial.

Notice that for all $g \in B_m(1; S)$, the shift $gB_m(1; S)$ still contains $1 \in \Gamma$. Then we obtain a nontrivial relation $V(v_1, \ldots, v_s)$ among $v_1, \ldots, v_s$ of length at most $2m^2$.[2] We can write $V(v_1, \ldots, v_s) = W(h_1, \ldots, h_r) = 1$ where W is a nontrivial reduced word. Notice that $V(v_1, \ldots, v_s) = W(h_1, \ldots, h_r)$ represents a map in $PL_o(I)$, while $V(w_1, \ldots, w_s)$ represents a word in G_r which by our choices does not represent an identity element in G_r. Then, by Lemma 3.2, for some word $U = U(h_1, \ldots, h_r)$ and for all $x \in U((a_{r+1}, b_{r+1}))$ we have $W(x) \notin U((a_{r+1}, b_{r+1}))$, thus $W(x) \neq x$. Since m is arbitrary, we conclude that $girth(\Gamma) = \infty$. □

The main idea of the proof of Theorem 1.2 is described below. For any given q, k, in a suitable tower of sufficiently big height r, formed by PL-maps $\phi_1, \ldots, \phi_r, \phi_{r+1}$, one can find words $w_1, \ldots, w_k$ in the alphabet $\{\phi_1^{\pm 1}, \ldots, \phi_r^{\pm 1}\}$ such that the corresponding elements (let us denote them by $\overline{w_1}, \ldots, \overline{w_k}$) do not have a relation of length less than q. This is because, upon the action on the innermost base of the tower (an orbital of ϕ_{r+1}), a suitable tower behaves as if the maps generate a genuine copy of G_r, for elements in the ball of certain radius. The problem is how to find a finite generating set without a short relation among the generators, not just among *some* elements. For this, we pick up a tower with sufficient height such that the PL homeomorphisms forming this tower are supported in a very small interval. This interval can be made arbitrarily small; therefore, using irreducibility of Γ, one can push this support (interval) close enough to the end 0 of the interval $[0, 1]$ such that the new support Ω satisfies conditions (i)–(iii), namely, (i) any PL map β from the ball $B_m(1)$ still keeps Ω inside an interval $(0, \epsilon_0)$; (ii) the image of Ω by elements of $B_m(1)$ with different slopes at 0 are disjoint; and (iii) the images of Ω by elements of $B_m(1)$ with the same slope at 0 are the same. Then we pick up a generating set which involves the elements $\overline{w_1}, \ldots, \overline{w_k}$. By properties (i)–(iii), we again obtain a short relation among $\overline{w_1}, \ldots, \overline{w_k}$ thus a contradiction.

4. Girth of subgroups with hyperbolic-like elements

In this section, we will prove Theorem 1.5.

Let $d(\Gamma) = s$, and m be a natural number. We will find $s + 2$ generators of Γ such that there is no relation of length m or less in Γ in these generators. (since m is arbitrary, this proves that $girth(\Gamma) = \infty$).

Let $S = \{X_1, \ldots, X_s\}$ be a finite generating set of Γ, and S^* be the symmetrization of S, that is $S^* = \{X_1, \ldots, X_s, X_1^{-1}, \ldots, X_s^{-1}\}$. Let also $p_0 \in (0, 1)$ (one could take $p_0 = \frac{1}{2}$). We can find a natural number $N > 4m$ and a sequence $0 = c_0 < c_1 < c_2 < \ldots < c_{2N} < c_{2N+1} = 1$ such that the following three conditions are satisfied.

(i) $p_0 \in (c_N, c_{N+1})$;
(ii) for all $X \in S^*$ and $p \in \{p_0, c_1, \ldots, c_{2N}\}$, $X(p) \notin \{c_1, \ldots, c_{2N}\} \backslash \{p\}$;

[2]For all $1 \leq k \leq s$, both ω_k and η_k are products of commuting conjugate copies supported on disjoint shifts of Ω. We obtain our relation by considering the copies over the shift by the identity element, that is, over the original copy of Ω.

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(iii) $W(X_1,\ldots,X_s,Y_N)(p_0)\subset(c_1,c_{2N})$ for all reduced words W of the form $W_0Y_N^{n_1}W_1Y_N^{n_2}\ldots W_{k-1}Y_N^{n_k}W_k$ where Y_N is any orientation-preserving homeomorphism satisfying the conditions $Y_N(c_i)=c_{i+4}$, $1\leq i\leq 2N-4$, $n_1,\ldots,n_k\in\{-1,1\}$, and W_i is a reduced word in the alphabet $\{X_1^{\pm1},\ldots,X_s^{\pm1}\}$ of length L_i, $0\leq i\leq k$ where $\sum_{i=0}^{k}L_i\leq 2m$.

Notice that condition (iii) implies that

$$W(X_1,\ldots,X_s,Y_n)(p_0)\in(c_{1+4d},c_{2N-4d})$$

where $d=2m-\sum_{i=0}^{k}L_i$.

Let $\delta=\min_{0\leq i\leq 2N}|c_{i+1}-c_i|$ and $\epsilon<\frac{1}{8}\min\{\delta,|p_0-c_N|,|p_0-c_{N+1}|\}$ be such that $X(p-\epsilon,p+\epsilon)\cap(q-\epsilon,q+\epsilon)=\emptyset$ for all $X\in S^*$ and distinct $p,q\in\{p_0,c_1,\ldots,c_{2N}\}$. Then we can find a natural number $M>2m$ and elements $\gamma,\theta\in\Gamma$ such that:

(iv) $Fix(\gamma)=\{0,a_1,a_2,\ldots,a_N,1\}$, $Fix(\theta)=\{0,b_1,b_2,\ldots,b_N,1\}$ and for all $1\leq i\leq N$, the inequalities $|a_i-c_{2i-1}|<\epsilon$ and $|b_i-c_{2i}|<\epsilon$ hold;

(v) for all $n\geq M$, we have $\gamma^{\pm n}(U_\gamma)\subset V_\gamma$ where

$$U_\gamma=\bigsqcup_{0\leq i\leq N}(a_i+\epsilon,a_{i+1}-\epsilon),\ V_\gamma=\bigsqcup_{0\leq i\leq N+1}(a_i-\epsilon,a_i+\epsilon);$$

(vi) for all $n\geq M$, we have $\theta^{\pm n}(U_\theta)\subset V_\theta$ where

$$U_\theta=\bigsqcup_{0\leq i\leq N}(b_i+\epsilon,b_{i+1}-\epsilon),\ V_\theta=\bigsqcup_{0\leq i\leq N+1}(b_i-\epsilon,b_i+\epsilon).$$

It is straightforward to make all these arrangements. (Let us also clarify that we define $a_0=b_0=0$ and $a_{N+1}=b_{N+1}=1$.)

Notice that $p_0\in U_\gamma\cup U_\theta$ and $p_0\notin V_\gamma\cup V_\theta$. Now, let $r\geq 2M$, and $S'=\{\gamma^r,\theta^r,\gamma^{r^2}X_1\theta^{r^2},\ldots,\gamma^{sr^2}X_s\theta^{sr^2}\}$.

Then if W_0 is a nontrivial reduced word in these generators of length at most m, and if W' is any suffix of W_0 in the alphabet S', then because of (i)–(vi), we have $W'(p_0)\in(c_1,c_N)$.

Indeed, W' can be written as:

$$W'=\alpha_1X_{j_1}^{\epsilon_1}\alpha_2X_{j_2}^{\epsilon_2}\ldots\alpha_kX_{j_k}^{\epsilon_k}\alpha_{k+1}$$

where for all $1\leq i\leq k, j_i\in\{1,\ldots,s\},\epsilon_i\in\{-1,1\}$ and α_i, $1\leq i\leq k+1$ belong to the set:

$$\{\gamma^n:|n|>M\}\cup\{\theta^n:|n|>M\}\cup\{\gamma^m\theta^n:|m|,|n|>M\}\cup\{\theta^m\gamma^n:|m|,|n|>M\}.$$

From the re-writing W' as a suffix of W_0, we obtain that $k\leq m$. On the other hand, for all $1\leq i\leq k,\epsilon\in\{-1,1\}$ and $x\in(c_4,c_{N-4})$ we have the inequality $Y_N^{-1}(x)<\alpha_i^{\epsilon}(x)<Y_N(x)$. Then letting $W_i=X_{n_i}^{\epsilon_i}$, $1\leq i\leq k$, we obtain that $|W_i|=1$, $1\leq i\leq k$ in the alphabet S^*, hence $\sum_{i=1}^{k}|W_i|=k\leq m<\dfrac{N}{4}$.

By condition (iii), for every suffix W'' of W', we have $W''(p_0)\in(c_1,c_{2N})$, in fact, $W''(p_0)\in(c_{1+4d},c_{2N-4d})$ where $d=2m-L$ and L is the number of occurrences of $X_i^{\pm1}$, $1\leq i\leq s$ in W''. So, we have $W'(p_0)\in(c_{1+4d},c_{2N-4d})$ with $d=2m-k$.

Then, inductively (ping-pong argument), $W_0(p_0)\in V_\gamma\cup V_\theta$; more precisely, $W_0(p_0)\in V_\gamma$ if W_0 starts with $\gamma^{\pm1}$ and $W_0(p_0)\in V_\theta$ if W_0 starts with $\theta^{\pm1}$. Then, $W_0(p_0)\neq p_0$, hence $W_0\neq 1$. □

Remark 4.1. *The assumptions of Theorem 1.5 can be weakened significantly (at the expense of making the statement more technical).*

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The proof of Theorem 1.5 uses the well-known “ping-pong idea” where one designs a certain “ping-pong table” which could involve two or more sets (in our case U_γ and U_θ), and (in one of the versions) the point (in our case p_0) taken outside of these sets arrives to one of these sets and then either jumps from one to another never returning back to its original position or in a slightly more subtle version as in our case, even if it leaves $U_\gamma \cup U_\theta$ (in our case this might happen by the action of $X \in S^*$), then immediately returns back to it. In condition (i) of our proof, p_0 is taken in the middle interval just for convenience. We are following the orbit of it by the consecutive suffixes of W_0. Notice that by our choice of r, there will not be any occurrence of $X_i^{\pm 1}X_j^{\pm 1}$ in W_0; moreover, the exponents of both γ and θ are sufficiently big. By condition (ii) and the choice of ϵ, our point stays away from $Fix(\gamma) \cup Fix(\theta)$ whenever $X \in S^*$ is applied. It may get close to $Fix(\gamma)$, then hits U_γ and is immediately taken away from it by $X \in S^*$ or by sufficiently big powers of θ. Similar effect happens when we get close to $Fix(\theta)$. Condition (iii) guarantees that we stay in the field of action where arrangements are suitable.

Remark 4.2. *From the proof we see that one can state a much more general theorem for the girth of groups acting on metric spaces by homeomorphisms. For every non-elementary word hyperbolic group we do have such a theorem indeed (see Theorem 2.6, [2], where the metric space is the boundary of the group, and for every hyperbolic element we have one attracting and one repelling point). In our case, the metric space is typically non-compact (in the case of Theorem 1.5, the metric space is the non-compact space* $(0, 1) \cong \mathbb{R}$*), and the “hyperbolic-like” elements have several points (instead of two) which are “attractive-repelling like” within “certain compact subspace.”*

We would like to give a precise definition of a hyperbolic-like element.

Definition 4.3. *Let* X *be a Hausdorff topological space,* Γ *be a subgroup of* $Homeo(X)$ *generated by a finite subset* $S \subseteq \Gamma$, $S^\star = S \cup S^{-1} \cup \{1\}$, $z \in X$, $m \in \mathbb{N}$, *and* $\gamma \in \Gamma$. *We say* γ *is* (S, z, m)*-hyperbolic-like if there exists a chain* $\Omega_0 \subset \Omega_1 \subset \ldots \subset \Omega_m$ *of finite subsets of* X *such that*

(i) $\Omega_0 = \{z\}$;
(ii) $s(\Omega_m) \cap \Omega_m = \emptyset, \forall s \in S^\star \backslash \{1\}$;
(iii) for all $x \in (S^\star \backslash \{1\})\Omega_k, 0 \leq k \leq m-1$, *there exist distinct* $p_a, p_r \in \Omega_{k+1}$ *such that for all disjoint open neighborhoods* U_{p_a}, U_{p_r} *of* p_a *and* p_r, *respectively, there exist* $M \in \mathbb{N}$ *such that for all* $n \geq M$, $\gamma^n(x) \in U_{p_a}, \gamma^{-n}(x) \in U_{p_r}$.

The proof of the following theorem utilizes the main idea of the proof of Theorem 1.5.

Theorem 4.4. *Let* X *be a Hausdorff space,* $z \in X$, Γ *be a finitely generated subgroup of Homeo* (X), S *be a finite generating set of* Γ. *Assume that for all natural m. there exists an* (S, z, m)*-hyperbolic-like element of* Γ. *Then girth*$(\Gamma) = \infty$.

Proof. Without loss of generality, we may assume that $1 \notin S$. Since X is Hausdorff, the attractive and repelling points p_a, p_r in condition (iii) are unique. By condition (ii), we also have $p_a, p_r \notin (S^\star \backslash \{1\})\Omega_k, 0 \leq k \leq m-1$. Then we can claim that there exists a natural number M such that for all $0 \leq k \leq m-1$, there exist unique and distinct $p_a^{(k)}, p_r^{(k)} \in \Omega_{k+1}$ and disjoint open neighborhoods $U_{p_a^{(k)}}, U_{p_r^{(k)}}$ of $p_a^{(k)}$ and $p_r^{(k)}$, respectively, such that for all $n > M$ and for all $x \in (S^\star \backslash \{1\})\Omega_k$, $\gamma^n(x) \in U_{p_a^{(k)}}, \gamma^{-n}(x) \in U_{p_r^{(k)}}$; moreover, $(U_{p_a^{(k)}} \sqcup U_{p_r^{(k)}}) \cap (S^\star \backslash \{1\})\Omega_k = \emptyset$. Then for sufficiently big n, there is no relation of length less than $m-2$ among the elements of the generating set $\{\gamma, \gamma^n\gamma_1\gamma^n, \gamma^{2n}\gamma_2\gamma^{2n}, \ldots, \gamma^{sn}\gamma_s\gamma^{sn}\}$ where $S = \{\gamma_1, \ldots, \gamma_s\}$.

Indeed, such a relation would be of the form:

$$W = \gamma^{n_1}\delta_1\gamma^{n_2}\delta_2 \ldots \gamma^{n_r}\delta_r\gamma^{n_{r+1}}$$

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where $r \leq m-2$, $|n_i| \geq n-m$, $1 \leq i \leq r+1$, and $\delta_i \in S^\star \backslash \{1\}$, $1 \leq i \leq r$. Let $y \in (S^\star \backslash \{1\})z$. Then for sufficiently big n, we have $W(y) \in U_{p_a^{(r+2)}} \sqcup U_{p_r^{(r+2)}}$. Hence, $W(y) \neq y$. Since m is arbitrary, we conclude that $girth(\Gamma) = \infty$. □

Remark 4.5. *Theorem 4.4 generalizes Theorem 2.1 from [2] which states that any finitely generated non-elementary subgroup of a word hyperbolic group has infinite girth.*

Remark 4.6. *It is interesting that the group F (the standard representation of it in $PL_o(I)$) is very rich with hyperbolic-like elements, for the standard finite generating set S of F. It is not known to us if the same can be true for a faithful representation of an elementary amenable subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$.

We will borrow the following definition from [3].

Definition 4.7. *Let Γ be a finitely generated group, $d(\Gamma)$ be the minimal number of a generating set of Γ and $k \geq d(\Gamma)$ be a positive integer. Then, we define* $girth_k(\Gamma) = \sup_{\langle S \rangle = \Gamma, |S| \leq k} girth(\Gamma, S)$.

While proving Theorem 1.2, we indeed proved a bit more, namely, for any non-solvable finitely generated group Γ of $PL_o(I)$, we proved that $girth_{2d}(\Gamma) = \infty$ where $d = d(\Gamma)$. Also, in the proof of Theorem 1.5, we indeed proved that $girth_{d+2}(\Gamma) = \infty$. With a slightly different argument, one can improve this result showing that $girth_{d+1}(\Gamma) = \infty$; and in Theorem 4.4, one can prove that $girth_{|S|+1}(\Gamma) = \infty$.

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