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Extensions of Continuous and Lipschitz Functions

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Abstract. We show a result slightly more general than the following. Let K be a compact Hausdorff space, F a closed subset of K, and d a lower semi-continuous metric on K. Then each continuous function f on F which is Lipschitz in d admits a continuous extension on K which is Lipschitz in d. The extension has the same supremum norm and the same Lipschitz constant.

As a corollary we get that a Banach space X is reflexive if and only if each bounded, weakly continuous and norm Lipschitz function defined on a weakly closed subset of X admits a weakly continuous, norm Lipschitz extension defined on the entire space X.

1 Introduction

The classical theorem of Tietze and Urysohn says that given a continuous function f on a closed subset F of a normal space T, there is a continuous extension $\tilde{f}$ of f to all of T so that $\inf_F f \le \tilde{f} \le \sup_F f$. Kirszbraun's theorem ensures that any Lipschitz function defined on a subset of a metric space M can be extended to a Lipschitz function on M with the same Lipschitz constant (see *e.g.*, [WW]). Given a normal space (T,τ) with some metric d on it, we examine when it is possible to extend every τ-continuous function Lipschitz in d defined on a τ-closed subset of T to a τ-continuous function Lipschitz in d defined on the entire space T. We show that every bounded, τ-continuous function Lipschitz in d defined on a τ-closed subset of T can be extended to a τ-continuous function Lipschitz in d defined on the entire space T with the same supremum and Lipschitz norm if and only if for each τ-closed subset F of T and $\varepsilon > 0$ the set

$$\{x \in T : d\text{-dist}(F,x) \le \varepsilon\}$$

is τ-closed. We give also an "in between" version of this result; strict one in the case when (T,τ) is countably paracompact. As a corollary we get that if (K,τ) is a compact Hausdorff space, d a lower semi-continuous metric on K, F a τ-closed subset of K, $c > 0$ and f a τ-continuous function on F which is c-Lipschitz in d then f admits a τ-continuous and c-Lipschitz extension $\tilde{f}$ on K such that $\inf_F f \le \tilde{f} \le \sup_F f$. A special case of this result with f taking only values 0 and 1 and the extension $\tilde{f}$ being "almost" c-Lipschitz appears in [GhMa] and [JNR].

As another corollary we get that each bounded, weak*-continuous and norm-Lipschitz function f defined on a weak*-closed subset of the dual X^* of a Banach space X admits

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a weak*-continuous norm-Lipschitz extension on X^* which has the same supremum and Lipschitz norm as f.

It is easy to see (*e.g.*, p. 214) that if each continuous Lipschitz function defined on a closed subset of a normal space (T,τ) with a metric d can be extended as above, then necessarily the metric d has to be lower semi-continuous with respect to τ. We give an example of a normal topological space (T,τ) with a lower semi-continuous metric on it (any separable nonreflexive Banach space with the weak topology and norm metric) and of a bounded, τ-continuous and 1-Lipschitz function f on a closed subset of T such that no τ-continuous extension of f is c-Lipschitz for any $c > 0$. Namely, we show that if X is a nonreflexive Banach space, there exists a weakly closed subset F of the unit ball B and a weakly continuous, norm Lipschitz function f on F, such that no weakly continuous extension of f on B is norm Lipschitz. Thus we get that a Banach space X is reflexive if and only if each bounded, weakly continuous and norm Lipschitz function defined on a weakly closed subset of X admits a weakly continuous, norm Lipschitz extension defined on the entire space X.

The functions in the hypotheses of Tietze-Urysohn and Kirszbraun's theorems do not have to be bounded; in our setting, they do have to be bounded. We give an example of an unbounded, weakly continuous and norm-Lipschitz function f defined on a weakly closed subset of the separable Hilbert space ℓ_2 such that no weakly-continuous extension of f on ℓ_2 is c-Lipschitz for any $c > 0$.

We consider only Hausdorff topological spaces. In the following, if (T,τ) is a topological space and d is some metric on T, if we do not specify which topology we mean, we always mean the topology τ, not the one defined on T by the metric d.

2 Extensions

Let X be a set and d a not necessarily symmetric pseudometric on X. By this we mean that $d\colon X\times X\to\Re$, $d\geq 0$, $d(x,x)=0$, $d(x,y)\leq d(x,z)+d(z,y)$ for all $x,y,z\in X$, but not necessarily $d(x,y)=d(y,x)$. If $c>0$, we say that a function $f\colon X\to\Re$ is c-Lipschitz in d if $f(x)-f(y)\leq c\,d(x,y)$, for all $x,y\in X$. If Y and Z are subsets of X, then

$$d\text{-dist}(Y,Z)=\inf\{d(y,z): y\in Y, z\in Z\},$$

whereas

$$d\text{-dist}(Z,Y)=\inf\{d(z,y): y\in Y, z\in Z\}.$$

By a slight abuse of notation we denote for $A\subset X$ and $\alpha>0$

$$d\text{-}B(A,\alpha)=\{x\in X: d\text{-dist}(A,x)\leq\alpha\},$$
$$d\text{-}B(\alpha,A)=\{x\in X: d\text{-dist}(x,A)\leq\alpha\}.$$

The specification "d-" will sometimes be omitted.

Suppose now that (X,τ) is a topological space, d is a (nonsymmetric) pseudometric on X and $f\colon X\to\Re$ is τ-continuous and 1-Lipschitz in d. Then the function $d'\colon X\times X\to\Re$ defined as $d'(x,y)=d(x,y)-\bigl(f(x)-f(y)\bigr)$ is clearly also a nonsymmetric pseudometric on X. Suppose that d has the property that if $A\subset X$ is τ-closed and $\alpha>0$ then both

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the sets d-$B(A,\alpha)$ and d-$B(\alpha,A)$ are τ-closed. Then d' also has this property. Indeed, let $x \in X \setminus d'$-$B(A,\alpha)$ be arbitrary. Then there is some $\varepsilon > 0$ so that $d'(a,x) = d(a,x) - \big(f(a) - f(x)\big) > \alpha + \varepsilon$ for all $a \in A$. Choose an open set $U_1 \subset X$ so that $x \in U_1$ and $|f(x) - f(y)| < \varepsilon/4$ for each $y \in U_1$. Let $\beta = \sup_{a\in A}\{0, f(a) - f(x) + \alpha + \varepsilon/2\}$. If $\beta = 0$ put $U_2 = X \setminus A$, otherwise let $U_2 = X \setminus d$-$B(A,\beta)$; in both cases $x \in U_2$. Let $y \in U = U_1 \cap U_2$ and $a \in A$ be arbitrary. Then

$$\begin{aligned} d'(a,y) = d(a,y) - \big(f(a) - f(y)\big) &\geq \beta - \big(f(a) - f(x)\big) - \big(f(x) - f(y)\big) \\ &> \alpha + \varepsilon/4. \end{aligned}$$

This means that $U \cap d'$-$B(A,\alpha) = \varnothing$, and the set d'-$B(A,\alpha)$ is closed. Similarly we get that the set d'-$B(\alpha,A)$ is closed.

The following Urysohn-like lemma is a mild extension of a result contained in [JNR]; we give only an outline of the proof. We use the following elementary property of F_σ sets: let X be a normal space and $A, B \subset X$ be F_σ sets such that $\bar{A} \cap B = A \cap \bar{B} = \varnothing$. Then there exists an open set $U \subset X$ so that $A \subset U$ and $\bar{U} \cap B = \varnothing$.

Lemma 2.1 *Let (X,τ) be a normal space and d be a (nonsymmetric) pseudometric on X with the property that if $A \subset X$ is τ-closed and $\alpha > 0$ then both the sets d-$B(A,\alpha)$ and d-$B(\alpha,A)$ are τ-closed. Suppose F_0 and F_1 are τ-closed disjoint nonempty subsets of X with*

$$d(x_1,x_0) \geq 1 \quad \text{for } x_1 \in F_1 \text{ and } x_0 \in F_0.$$

Then there exists $f\colon X \to [0,1]$ continuous in τ and 1-Lipschitz in d, taking the value 0 on F_0 and the value 1 on F_1.

Proof First observe that if $F \subset X$ is closed and $\alpha > 0$ then the set

$$\{x \in X : \operatorname{dist}(F,x) < \alpha\} = \bigcup_{\frac{1}{n}<\alpha} B\Big(F, \alpha - \frac{1}{n}\Big),$$

hence it is an F_σ set. Similarly the set $\{x \in X : \operatorname{dist}(x,F) < \alpha\}$ is F_σ.

Let Q be the set of all rational numbers in $(0,1)$. Enumerate $Q \cup \{0,1\}$ so that $r_0 = 0, r_1 = 1, r_2, \ldots$. We use the convention that $U_0 = \bar{U}_0 = F_0$ (this means that unlike the other U's U_0 is a closed set; it can have even empty interior) and $U_1 = X \setminus F_1$. We construct a family of open sets $\{U_r : r \in Q\}$ in X so that:

(i) for $s, t \in Q \cup \{0,1\}$, $s < t$, and any $x \in \bar{U}_s$, $y \in X \setminus U_t$, we have $d(y,x) \geq t - s$.

Suppose that for some $n \geq 1$, the sets U_{r_i}, $0 \leq i \leq n$, have been chosen so that (i) holds for all choices of s, t from $\{r_0, r_1, \ldots, r_n\}$. The set $U_{r_{n+1}}$ will be chosen in the following way. Write $r = r_{n+1}$ and

$$\begin{aligned} S &= \{r_j : 0 \leq j \leq n, r_j < r\}, \\ T &= \{r_j : 0 \leq j \leq n, r < r_j\}. \end{aligned}$$

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Put

$$A = \bigcup_{s \in S} \{x \in X : \operatorname{dist}(x, \bar{U}_s) < r - s\}$$

$$A' = \bigcup_{s \in S} B(r - s, \bar{U}_s)$$

$$B = \bigcup_{t \in T} \{x \in X : \operatorname{dist}(X \setminus U_t, x) < t - r\}$$

$$B' = \bigcup_{t \in T} B(X \setminus U_t, t - r).$$

Both A and B are F_σ sets; the sets A' and B' are closed and $A \subset A'$ and $B \subset B'$. By (i) we have that $A' \cap B = A \cap B' = \varnothing$. Therefore there exists an open set U_r so that

$$A \subset U_r \quad \text{and} \quad \bar{U}_r \cap B = \varnothing.$$

If we define a function f on X by taking f to be 1 on F_1, and

$$f(x) = \inf\{r : x \in U_r, r \in Q\} \quad \text{for } x \in U_1$$

then f is continuous by the proof of Urysohn's lemma [K, p. 114]. If $x, y \in X$ and $f(x) = a < b = f(y)$ then for all $a < s < t < b$ we have $x \in U_s$ and $y \in X \setminus U_t$. Hence

$$d(y, x) \geq \operatorname{dist}(X \setminus U_t, U_s) \geq s - t,$$

and $d(y, x) \geq b - a = f(y) - f(x)$, which means that f is 1-Lipschitz in d. ■

Theorem 2.2 *Let (K, τ) be a normal topological space, and d be a metric on K such that the set $B(A, \varepsilon)$ is τ-closed for each τ-closed $A \subset K$ and $\varepsilon > 0$; $c > 0$. Let $g \leq h$ be bounded functions on K so that $g(x) - h(y) \leq c\, d(x, y)$ for each $x, y \in K$. If g is upper semi-continuous in τ, and h is lower semi-continuous in τ then there exists a function f on K which is τ-continuous, c-Lipschitz in d, and $g \leq f \leq h$.*

Proof By adding a constant and multiplying by a constant of g and h we can suppose that $-1 \leq g \leq h \leq 1$; by multiplying the metric by a constant we can suppose that $c = 1$. Put $g_0 = g$, $h_0 = h$, and $d_0 = d$. As in the proof of Tietze's theorem we proceed by induction. Suppose that d_k is a (nonsymmetric) pseudometric on K satisfying the assumptions of Lemma 2.1 and $g_k \leq h_k$ are functions on K so that $g_k \leq 2^k 3^{-k}$, $h_k \geq -2^k 3^{-k}$, $g_k(x) - h_k(y) \leq d_k(x, y)$ for each $x, y \in K$; g_k is upper semi-continuous in τ, and h_k is lower semi-continuous in τ. Put

$$G_k = \left\{ x \in K : g_k(x) \geq \frac{2^k}{3^{k+1}} \right\}$$

$$H_k = \left\{ x \in K : h_k(x) \leq -\frac{2^k}{3^{k+1}} \right\}.$$

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It is $d_k(x,y) \geq 2^{k+1}3^{-(k+1)}$ for any $x \in G_k$ and $y \in H_k$ and by Lemma 2.1 there exists a τ-continuous function ψ_k which is 1-Lipschitz in d_k, $-2^k3^{-(k+1)} \leq \psi_k \leq 2^k3^{-(k+1)}$, $\psi_k = -2^k3^{-(k+1)}$ on H_k and $\psi_k = 2^k3^{-(k+1)}$ on G_k. (If one of the sets G_k, H_k, say G_k, is empty, we put $\psi_k = -2^k3^{-(k+1)}$; if $G_k = H_k = \varnothing$, we set $\psi_k = 0$.) Put $g_{k+1} = g_k - \psi_k$, $h_{k+1} = h_k - \psi_k$, and $d_{k+1}(x,y) = d_k(x,y) - \big(\psi_k(x) - \psi_k(y)\big)$ for $x, y \in K$. By the remarks preceding Lemma 2.1, d_{k+1} is a pseudometric which satisfies the assumptions of Lemma 2.1. Clearly, $g_{k+1} \leq h_{k+1}$, $g_{k+1} \leq 2^{k+1}3^{-(k+1)}$, $h_{k+1} \geq -2^{k+1}3^{-(k+1)}$, $g_{k+1}(x) - h_{k+1}(y) \leq d_{k+1}(x,y)$ for each $x, y \in K$; g_{k+1} is upper semi-continuous in τ, and h_{k+1} is lower semi-continuous in τ. Put $f = \sum_{k=0}^{\infty} \psi_k$. Then f is well defined and τ-continuous; $-1 \leq f \leq 1$. From the construction it follows that

$$g - \sum_{i=0}^{k} \psi_i = g_{k+1} \leq 2^{k+1}3^{-(k+1)} \quad \text{and}$$

$$h - \sum_{i=0}^{k} \psi_i = h_{k+1} \geq -2^{k+1}3^{-(k+1)},$$

for $k \in \mathbb{N}$, hence $g \leq f \leq h$. By induction we have also that $d_{k+1}(x,y) = d(x,y) - \sum_{i=0}^{k}\big(\psi_i(x) - \psi_i(y)\big)$ for $k \in \mathbb{N}$ and $x, y \in K$. Since

$$\psi_{k+1}(x) - \psi_{k+1}(y) \leq d_{k+1}(x,y) = d(x,y) - \sum_{i=0}^{k}\big(\psi_i(x) - \psi_i(y)\big)$$

we have

$$\sum_{i=0}^{k+1}\big(\psi_i(x) - \psi_i(y)\big) \leq d(x,y)$$

for $k \in \mathbb{N}$ and $x, y \in K$ which means that f is 1-Lipschitz in d. ■

If (K,τ) is a normal space and d is a discrete metric on K (that is $d(x,y) = 1$ if $x \neq y$), then d satisfies the assumptions of Theorem 2.2 and any function φ on K with $0 \leq \varphi \leq 1$ is 1-Lipschitz in d. Therefore by a theorem of Dowker and Katětov (see [E, p. 428]) if we wish to have sharp inequalities in Theorem 2.2 we have to assume that (K,τ) is countably paracompact. Also, we have to assume that both g and h are c-Lipschitz as the example of $c = 1$, $K = \{-1\} \cup (0,1]$, $g(-1) = 1$, $h(-1) = 2$, and $g(x) = 0$, $h(x) = x^2$ for $x \in (0,1]$ shows.

Proposition 2.3 *Let (K,τ) be normal and countably paracompact, and d be a metric on K such that the set $B(A,\varepsilon)$ is τ-closed for each τ-closed $A \subset K$ and $\varepsilon > 0$; $c > 0$. Let $g < h$ be bounded functions on K, both c-Lipschitz in d. If g is upper semi-continuous in τ, and h is lower semi-continuous in τ then there exists a function f on K which is τ-continuous, c-Lipschitz in d, and $g < f < h$.*

Proof First we show that there is a τ-continuous function f_1 on K which is c-Lipschitz in d and for which $g < f_1 \leq h$. Similarly one shows that there is f_2 with $g \leq f_2 < h$, and $f = \frac{1}{2}(f_1 + f_2)$ is the required function.

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For each pair of rational numbers $r < s$ put

$$U_{r,s} = \{x \in K : g(x) < r < s < h(x)\}.$$

The lower semi-continuity of g and h implies that each $U_{r,s}$ is open (possibly empty). Since $g < h$, $\mathcal{U} = \{U_{r,s}\}$ is a countable open cover of K. Let $\mathcal{V} = \{V_{r,s}\}$ be a closed cover of K with $V_{r,s} \subset U_{r,s}$; it exists since (K, τ) is countably paracompact (see *e.g.*, [E, p. 393]). Put

$$g_{r,s}(x) = \begin{cases} g(x), & \text{if } x \in X \setminus V_{r,s} \\ g(x) + s - r, & \text{if } x \in V_{r,s}. \end{cases}$$

Then $g \le g_{r,s} \le h$ and $g_{r,s} > g$ on $V_{r,s}$. If $\alpha \in \Re$, then

$$\{x \in K : g_{r,s}(x) \ge \alpha\} = \{x \in K : g(x) \ge \alpha\} \cup \{x \in V_{r,s} : g(x) \ge \alpha - (s - r)\};$$

since g is upper semi-continuous these sets are closed and $g_{r,s}$ is also upper semi-continuous. If $x, y \in K$ then $g_{r,s}(x) - h(y) \le h(x) - h(y) \le c\, d(x, y)$. By Theorem 2.2 there is a τ-continuous function $\varphi_{r,s}$ on K which is c-Lipschitz in d with $g_{r,s} \le \varphi_{r,s} \le h$. Re-index the functions φ by natural numbers and put $f_1 = \sum_{i=1}^{\infty} 2^{-i}\varphi_i$. Then f_1 is τ-continuous, c-Lipschitz in d and $g \le f_1 \le h$. Since $\mathcal{V}$ covers K it is even $g < f_1$ on K. ■

Theorem 2.4 *Let (K, τ) be a normal topological space, and d be a metric on K such that the set $B(A, \varepsilon)$ is τ-closed for each τ-closed $A \subset K$ and $\varepsilon > 0$; let $c > 0$. Let $F \subset K$ be closed, f be a bounded and τ-continuous function on F which is c-Lipschitz in d. Then there is a τ-continuous function $\tilde{f}$ on K such that $\tilde{f} = f$ on F, $\inf_F f \le \tilde{f} \le \sup_F f$, and $\tilde{f}$ is c-Lipschitz in d.*

Proof Define functions g and h on K so that $g = h = f$ on F, $g = \inf_F f$ on $K \setminus F$, and $h = \sup_F f$ on $K \setminus F$. It is easy to see that g and h satisfy the conditions of Theorem 2.2, hence there exists a continuous function $\tilde{f}$ defined on K which is c-Lipschitz in d and $g \le \tilde{f} \le h$. ■

There is a converse to the above theorem. Namely suppose there exists a closed set $A \subset K$ and $r > 0$ such that $B(A, r)$ is not closed. Choose some $z \in \overline{B(A, r)} \setminus B(A, r)$, and put $R = \operatorname{dist}(A, z)$. Then $r < R$ and the function

$$g(x) = \begin{cases} 0, & \text{if } x \in A \\ R, & \text{if } x = z. \end{cases}$$

is a continuous 1-Lipschitz function on the closed set $F = A \cup \{z\}$. Suppose g admits a continuous 1-Lipschitz extension f to K. If $u \in B(A, r)$, and $\varepsilon > 0$ then there exists $v \in A$ so that $d(u, v) < r + \varepsilon$, hence

$$f(u) = f(u) - f(v) \le d(u, v) < r + \varepsilon.$$

Since f is continuous, $f \le r$ on $\overline{B(A, r)}$, which is a contradiction.

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A metric d on a topological space K is *lower semi-continuous*, if d is lower semi-continuous as a real valued function on $K \times K$, that is, the set

$$\{(x, y) \in K \times K : d(x, y) \le \varepsilon\}$$

is closed for all $\varepsilon > 0$. Notice that the metric d in the previous theorem is necessarily lower semi-continuous. Indeed, given any two points $s, t \in K$, by Theorem 2.4 there exists a continuous function $f = f_{s,t}$ on K such that $0 \le f \le d(s, t)$, $f(s) = 0$, $f(t) = d(s, t)$, and f is 1-Lipschitz in d. If we put

$$\rho(s, t) = \sup\{|f_{u,v}(s) - f_{u,v}(t)| : u, v \in K\}$$

then clearly $d = \rho$ and ρ is lower semi-continuous on $K \times K$ as a pointwise supremum of a family of continuous functions. If K is a compact Hausdorff space, we get by the following corollary that a metric d on K is lower semi-continuous if and only if it has the property required in Theorem 2.4.

Corollary 2.5 *Let K be a compact Hausdorff space, d a lower semi-continuous metric on K, $F \subset K$ closed and $c > 0$. Let $g \in C(F)$ be c-Lipschitz in d. Then there exists $f \in C(K)$ such that $f = g$ on F, $\min_F g \le f \le \max_F g$, and f is c-Lipschitz in d.*

Proof Let $A \subset K$ be closed, and $\varepsilon > 0$. If $z \in K$ than $\operatorname{dist}(A, z) = \inf_{A \times \{z\}} d$, and since $A \times \{z\}$ is compact and d is lower semi-continuous, the infimum is attained. Hence

$$B(A, \varepsilon) = p_2\big((A \times K) \cap \{(x, y) \in K \times K : d(x, y) \le \varepsilon\}\big),$$

where p_2 is the projection on the second coordinate. Since A and K are compact and p_2 is continuous, the set $B(A, \varepsilon)$ is closed. ■

Corollary 2.6 *Let X be a Banach space and F a weak*-closed subset of the dual X^* of X; $c > 0$. Let g be a bounded, weak*-continuous function on F which is c-Lipschitz in the norm-metric on X^*. Then there exists a weak*-continuous function f on X^* such that $f = g$ on F, $\inf_F g \le f \le \sup_F g$, and f is c-Lipschitz in the norm-metric on X^*.*

Proof Since (X^*, weak^*) is σ-compact, it is Lindelöf. From the definition of the weak* topology it follows easily that it is regular. By a theorem of Tychonoff (see *e.g.*, [K, p. 113]) (X, weak^*) is normal. Let $A \subset X^*$ be weak*-closed and $\varepsilon > 0$. Observe that $B(A, \varepsilon) = A + B(0, \varepsilon)$; the latter set is closed since it is a sum of a weak*-closed set and of a weak*-compact set. Indeed, if $z \in X^*$ and $\operatorname{dist}(A, z) \le \varepsilon$, then $C = A \cap B(z, 2\varepsilon)$ is a nonempty weak*-compact set with $\operatorname{dist}(C, z) \le \varepsilon$. The function $h(x) = \|x - z\|$ is weak*-lower semi-continuous, hence it attains its minimum at some point $y \in C \subset A$. Then $\|y - z\| \le \varepsilon$, and $z \in \big(y + B(0, \varepsilon)\big)$. ■

3 Examples

As we have seen above, τ-lower semi-continuity of the metric d is a necessary condition for the conclusion of Theorem 2.4 to be valid. It is not sufficient, though; the next theorem shows that each separable nonreflexive Banach space equipped with the weak topology and norm metric provides an example. Indeed, the norm-metric on any Banach space is lower semi-continuous in the weak topology; weak topology is easily seen to be regular, separable Banach spaces are Lindelöf and therefore normal in the weak topology (see *e.g.*, [K, p. 113]).

Theorem 3.1 *Let X be a Banach space. Then X is not reflexive if and only if there exists a bounded, weakly closed subset F of X and a weakly continuous function g on F which is 1-Lipschitz in norm such that no continuous extension of g on X is c-Lipschitz for any $c > 0$.*

Proof If X is reflexive then X is a dual of X^* and the weak and weak* topology are the same; every weakly-continuous f which is Lipschitz in norm admits an extension by Corollary 2.6.

Suppose that X is not reflexive. Fix $0 < \delta < 1$. We will construct a weakly closed set $F_\delta \subset B(0,2)$ such that $\operatorname{dist}(F_\delta, 0) \geq \frac{1}{2}$, and $0 \in \overline{F_\delta + B(0,\delta)}^{\text{weak}}$. Recall that since X is nonreflexive by a result of James [Ja] there exists a sequence $\{u_n\}_{\mathbb{N}}$ in the unit ball of X so that for each $n \in N$

$$\operatorname{dist}\big(\operatorname{span}\{u_i\}_{i=1}^n, \operatorname{conv}\{u_i\}_{i=n+1}^\infty\big) > 1 - \tfrac{1}{3}\delta. \tag{1}$$

Put

$$F_\delta = \{u_j - (1-\delta)u_i : i, j \in \mathbb{N}, i < j\}.$$

Then clearly $F_\delta \subset B(0,2)$, and by (1) $\operatorname{dist}(F_\delta, 0) \geq \frac{1}{2}$. Let $z \in \overline{F_\delta}^{\text{weak}}$ be given. Then z is contained in the norm-closure of $\operatorname{span}\{u_i\}_{i=1}^\infty$. Choose $n \in \mathbb{N}$ so that

$$\operatorname{dist}\big(\operatorname{span}\{u_i\}_{i=1}^n, z\big) < \tfrac{1}{3}\delta,$$

and $v \in \operatorname{span}\{u_i\}_{i=1}^n$ so that $\|v - z\| < \frac{1}{3}\delta$. By the Hahn-Banach theorem choose z^* from the unit ball of X^* so that $z^* = 0$ on $\operatorname{span}\{u_i\}_{i=1}^n$ and

$$\langle z^*, x\rangle > 1 - \tfrac{1}{3}\delta$$

for all $x \in \operatorname{conv}\{u_i\}_{i=n+1}^\infty$. Then for each for each $i, j \in \mathbb{N}$ such that $i < j$ and $n < j$

$$\begin{aligned}\langle z^*, u_j - (1-\delta)u_i - z\rangle &= \langle z^*, u_j\rangle - (1-\delta)\langle z^*, u_i\rangle + \langle z^*, v - z\rangle - \langle z^*, v\rangle \\ &> 1 - \tfrac{1}{3}\delta - (1-\delta) - \tfrac{1}{3}\delta - 0 = \tfrac{1}{3}\delta.\end{aligned}$$

Since the set $\{u_j - (1-\delta)u_i : i, j \in \mathbb{N}, i < j \leq n\}$ is finite, $z \in F_\delta$.

To show that $0 \in \overline{F_\delta + B(0,\delta)}^{\text{weak}}$, let $x_1^*, \ldots, x_n^*$ in the unit ball of X^* and $\varepsilon > 0$ be given. Observe that

$$\{u_j - u_i : i, j \in \mathbb{N}, i < j\} \subset F_\delta + B(0,\delta).$$

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Since for each $1 \le l \le n$ the sequence $(\langle x_l^*, u_i\rangle)_{i\in\mathbb{N}}$ is bounded, there exist $a_1, \dots, a_n \in \Re$ and a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ of $(u_i)_{i\in\mathbb{N}}$ such that

$$|\langle x_l^*, u_{k_i}\rangle - a_l| < \tfrac{\varepsilon}{2}$$

for each $1 \le l \le n$ and $i \in \mathbb{N}$. Consequently

$$|\langle x_l^*, u_{k_2} - u_{k_1}\rangle| < \varepsilon$$

for each $1 \le l \le n$, and 0 is in the weak closure of $F_\delta + B(0,\delta)$.

Now choose a bounded sequence (z_n) in X such that

$$\operatorname{dist}\big(\operatorname{span}\{z_i\}_{i=1}^{n-1}, \operatorname{conv}\{z_i\}_{i=n}^{\infty}\big) > 5$$

for each $n \in \mathbb{N}$. Put $F = \{z_n\}_{n=2}^{\infty} \cup \bigcup_{n=2}^{\infty} F_{\frac{1}{n}} + z_n$. The set $\bigcup_{n=2}^{\infty} F_{\frac{1}{n}} + z_n$ is weakly closed since each $F_{\frac{1}{n}} + z_n$ is weakly closed and

$$\operatorname{dist}\Big(\operatorname{conv}\bigcup_{i=2}^{n-1} F_{\frac{1}{i}} + z_i, \operatorname{conv}\bigcup_{i=n}^{\infty} F_{\frac{1}{i}} + z_i\Big) \ge 1$$

for each $n \ge 3$. Since $\{z_n\}_{\mathbb{N}}$ is weakly closed, F is weakly closed as well. Define

$$g(x) = \begin{cases} 0, & \text{if } x \in \bigcup_{n=2}^{\infty}(F_{\frac{1}{n}} + z_n) \\ \frac{1}{2}, & \text{if } x \in \{z_n\}_{n=2}^{\infty}. \end{cases}$$

It is readily seen that g is a weakly continuous and 1-Lipschitz function. Suppose $n \in \mathbb{N}$ and f is a weakly continuous, n-Lipschitz extension of g on X. Let $x \in B(z_{4n} + F_{\frac{1}{4n}}, \frac{1}{4n})$ be arbitrary; choose $y \in (z_{4n} + F_{\frac{1}{4n}})$ so that $\|x - y\| \le \frac{1}{3n}$. Then

$$f(x) = f(x) - f(y) \le n\|x - y\| \le \tfrac{1}{3}.$$

Hence $f \le \frac{1}{3}$ on $B(z_{4n}+F_{\frac{1}{4n}}, \frac{1}{4n})$, and since $z_{4n} \in \overline{B(z_{4n} + F_{\frac{1}{4n}}, \frac{1}{4n})}^{\text{weak}}$, this is a contradiction. ■

The following example shows that unlike Tietze-Urysohn and Kirszbraun's theorems, the function in the hypothesis of Theorem 2.4 has to be bounded.

Example 3.2 There exists a weakly closed subset F of the Hilbert space ℓ_2 and an unbounded, weakly continuous function g on F which is 1-Lipschitz in norm, such that no continuous extension of g on ℓ_2 is c-Lipschitz for any $c > 0$.

Let $(e_i)_{\mathbb{N}_o}$ be the canonical basis of ℓ_2. Define

$$x^n = n^{\frac{1}{4}} e_o + n^{\frac{1}{2}} e_n$$
$$y^n = n^{\frac{1}{2}} e_n;$$

observe that zero is in the weak closure of the set $\{y^k\}_{k\geq n}$ for each $n \in \mathbb{N}$. Indeed if $\alpha = (\alpha_i) \in \ell_2$, $\varepsilon > 0$ then there exists $k \geq n$ so that $|\langle y^k, \alpha\rangle| = |k^{\frac{1}{2}}\alpha_k| < \varepsilon$: otherwise $|\alpha_k| \geq \varepsilon k^{-\frac{1}{2}}$ for $k \geq n$ and $(\alpha_i) \notin \ell_2$. Similarly one can argue for finitely many α's.

Put $F = \{x^n\}_{\mathbb{N}}$, $F_n = \{x^m : n \leq m\}$. Since $\lim_{m\to\infty} x_o^m = \infty$, each of the sets F_n is weakly closed, and the function $g\colon F \to \Re$ defined by $g(x^n) = n^{\frac{1}{2}}$ is continuous. Since for $n > m$

$$|g(x^n) - g(x^m)| = n^{\frac{1}{2}} - m^{\frac{1}{2}} \leq (n+m)^{\frac{1}{2}} \leq \big((n^{\frac{1}{4}} - m^{\frac{1}{4}})^2 + n + m\big)^{\frac{1}{2}} = \|x^n - x^m\|,$$

the function g is 1-Lipschitz. Suppose f is a weakly continuous, c-Lipschitz extension of g on ℓ_2; denote $a = f(0)$. Choose $n \in \mathbb{N}$ so that if $m \geq n$ then

$$m^{\frac{1}{2}} - cm^{\frac{1}{4}} \geq a + 1.$$

Then for $m \geq n$

$$f(y^m) \geq f(x^m) - c\|y^m - x^m\| = m^{\frac{1}{2}} - cm^{\frac{1}{4}} \geq a + 1.$$

Since zero is in the closure of the set $\{y^m\}_{m\geq n}$, $f(0) \geq a + 1$ which is a contradiction.

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