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MULTIPLE POSITIVE SOLUTIONS OF SINGULAR POSITONE DIRICHLET PROBLEMS WITH DERIVATIVE DEPENDENCE*

BAOQIANG YAN
Department of Mathematics, Shandong Normal University, Ji-nan, 250014, P.R. China

DONAL O'REGAN
Department of Mathematics, National University of Ireland, Galway, Ireland

and RAVI P. AGARWAL
Department of Mathematical Science, Florida Institute of Technology, Melbourne, Florida 32901, USA

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Abstract. The existence of multiple positive solutions is presented for the singular Dirichlet boundary value problems

$$\begin{cases} x'' + \Phi(t)f(t, x(t), |x'(t)|) = 0, \\ x(0) = 0, \quad x(1) = 0, \end{cases}$$

using the fixed point index; here f may be singular at $x = 0$ and $x' = 0$.

1. Introduction. In [**11**], S. Staněk considered

$$\begin{cases} x'' + \mu q(t)f(t, x, x') = 0, \\ x(0) = 0, \ x(T) = 0, \end{cases}$$

where $q(t) > 0$ for $t \in [0, T]$, $A > 0$ is a constant and $f(t, x, x') > 0$ is continuous on $[0, T] \times (0, +\infty) \times [-\frac{2A}{T}, 0) \cup (0, \frac{2A}{T}]$ and may be singular at $x = 0$, $x = A$ and $x' = 0$. Now x is a solution of the above equation if $x(0) = x(T) = 0$ and x satisfies $x''(t) + \mu q(t)f(t, x(t), x'(t)) = 0$ for all $t \in \{t \in (0, T) : x'(t) \neq 0\}$. Existence of one solution was established in [**11**] using regularity and sequential techniques. In [**1**] existence of the above equation was established when f was singular at $x = 0$ but not at $x' = 0$. However in [**1**] f was allowed to change sign. Various other existence results for $x'' + f(t, x, x') = 0$ were established [**7, 9–11**] when f is singular at $x' = 0$. If $f(t, x, x') = f(t, x)$ is singular at $x = 0$ there are many results on the existence of multiple positive solutions to $x'' + f(t, x) = 0$ (see [**2–4**], [**11**]). However to date there are only a few results [**6, 12**] on the existence of multiple solutions to $x'' + f(t, x, |x'|) = 0$ when f has a derivative dependence. The goal of this paper is to attempt to fill this gap in the literature.

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In this paper we consider the singular Dirichlet boundary value problems

$$\begin{cases} x'' + \Phi(t)f(t, x, |x'|) = 0, \\ x(0) = 0, x(1) = 0, \end{cases} \tag{1.1}$$

where f may be singular at $x = 0$ and $x' = 0$. The definition of a solution to (1.1) is as above (as in [**11**]). There are main three sections in our paper. In Section 2, we define a special Banach space, construct a special cone in it and give its properties. In Section 3, using the theory of fixed point index, we present a new result on the existence of multiple solutions to (1.1) when f is singular at $x' = 0$ and not singular at $x = 0$. In Section 4 when f is singular at $x = 0$ and $x' = 0$ we establish the existence of multiple positive solutions to (1.1).

2. Preliminaries. Let $q(t) = t(1-t)$, $t \in [0, 1]$ and $C_q^1[0, 1] = \{x : [0, 1] \to R|\ x$ is continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$ with $\sup_{t\in(0,1)} q(t)|x'(t)| = \sup_{t\in(0,1)} t(1-t)|x'(t)| < +\infty\}$. For $x \in C_q^1[0, 1]$, define $\|x\| = \max\{\|x\|_1, \|x\|_2\}$, where $\|x\|_1 = \max_{t\in[0,1]} |x(t)|$ and $\|x\|_2 = \sup_{t\in(0,1)} t(1-t)|x'(t)|$.

LEMMA 2.1. *$C_q^1[0, 1]$ is a Banach space. For any $x \in C_q^1[0, 1]$, $|x'(t)| \leq \frac{\|x\|}{t(1-t)}$, $t \in (0, 1)$.*

Proof. Clearly $C_q^1[0, 1]$ is a linear space with zero element $\theta(t) \equiv 0$ for $t \in [0, 1]$ and it is also easy to check that $\|\cdot\|$ is a norm on $C_q^1[0, 1]$.

We show that $C_q^1[0, 1]$ is complete. Assume $\{x_n\}_{n=1}^{\infty} \subseteq C_q^1[0, 1]$ is a Cauchy sequence, i.e., for any $\varepsilon > 0$, there exists an $N > 0$ such that $\|x_n - x_m\| < \varepsilon$ for all $n > N, m > N$. Since $\|x_n - x_m\|_1 \leq \|x_n - x_m\|$, $\{x_n\} \subseteq C[0, 1]$ is a Cauchy sequence with

$$\lim_{n\to+\infty} \|x_n - x_0\|_1 = 0, \ \ x_0 \in C[0, 1]. \tag{2.1}$$

Also for any given $\delta > 0$, one has

$$\begin{aligned} \delta(1-\delta) \max_{t\in[\delta,1-\delta]} |x_n'(t) - x_m'(t)| &\leq \max_{t\in[\delta,1-\delta]} t(1-t)|x_n'(t) - x_m'(t)| \\ &\leq \sup_{t\in(0,1)} t(1-t)|x_n'(t) - x_m'(t)| = \|x_n - x_m\|_2 \leq \|x_n - x_m\|, \end{aligned}$$

which means $\max_{t\in[\delta,1-\delta]} |x_n'(t) - x_m'(t)| \leq \frac{1}{\delta(1-\delta)}\|x_n - x_m\|$, $\forall n > N, m > N$.

Consequently, x_0 is continuously differentiable on $[\delta, 1-\delta]$ with $\lim_{n\to+\infty} x_n'(t) = x_0'(t)$ uniformly on $[\delta, 1-\delta]$. Since δ is arbitrary, x_0 is continuously differentiable on $(0, 1)$. Now $\sup_{t\in(0,1)} t(1-t)|x_n'(t) - x_m'(t)| \leq \|x_n - x_m\| < \varepsilon$, so letting $m \to +\infty$ yields

$$\sup_{t\in(0,1)} t(1-t)|x_n'(t) - x_0'(t)| \leq \varepsilon. \tag{2.2}$$

Also from (2.1) and (2.2), one has $\lim_{n\to+\infty} \|x_n - x_0\| = 0$.

Finally from (2.2), it is easy to see that $x_0 \in C_q^1[0, 1]$. Hence, $C_q^1[0, 1]$ is a Banach space. If $x \in C_q^1[0, 1]$, then $\sup_{t\in(0,1)} t(1-t)|x'(t)| = \|x\|_2 \leq \|x\|$, which means $|x'(t)| \leq \frac{\|x\|}{t(1-t)}$ for all $t \in (0, 1)$. $\square$

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Let

$$G(t,s)=\begin{cases}(1-t)s, & 0\le s\le t\le 1\\ t(1-s), & 0\le t\le s\le 1\end{cases} \tag{2.3}$$

and

$$P=\left\{x\in C_q^1[0,1]: x(t)\ge t(1-t)\|x\|_1, \forall t\in[0,1] \text{ and } x\left(\tfrac{1}{2}\right)\ge \tfrac{1}{4}\|x\|_2\right\}. \tag{2.4}$$

The following lemmas are needed in Section 3 and Section 4.

Lemma 2.2 (see [**5**]). *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\overline{\Omega}\cap P\to P$ be continuous and compact. Suppose that $\lambda Ax\ne x$, $\forall x\in\partial\Omega\cap P$, $\lambda\in(0,1]$. Then*

$$i(A,\Omega\cap P,P)=1. \tag{2.5}$$

Lemma 2.3 (see [**5**]). *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\overline{\Omega}\cap P\to P$ be continuous and compact. Suppose that $Ax\not\le x$, $\forall x\in\partial\Omega\cap P$. Then*

$$i(A,\Omega\cap P,P)=0. \tag{2.6}$$

Lemma 2.4. *If $x\in P$ (defined as in (2.4)), then $\|x\|\le 4\|x\|_1$.*

Proof. If $x\in P$, one has $\|x\|_1=\max\{|x(t)|: t\in[0,1]\}\ge x(\frac{1}{2})\ge\frac{1}{4}\|x\|_2$. Then $\|x\|=\max\{\|x\|_1,\|x\|_2\}\le\max\{\|x\|_1,4x(\frac{1}{2})\}\le\max\{\|x\|_1,4\|x\|_1\}=4\|x\|_1$. □

Lemma 2.5. *Assume that $\overline{\Phi}\in C((0,1),R^+)$ with $\int_0^1\overline{\Phi}(t)\,dt<\infty$ and let $F(t)=\int_0^1 G(t,s)\overline{\Phi}(s)\,ds$. Then*

$$\begin{cases}F(t)\ge t(1-t)\max_{s\in[0,1]}F(s)=q(t)\|F\|_1, \quad \forall t\in[0,1]\\ \|F\|_2\le\|F\|_1\\ \|F\|_2=\sup_{t\in(0,1)}q(t)|F'(t)|\le 4F\left(\tfrac{1}{2}\right),\end{cases} \tag{2.7}$$

i.e., $F\in P$.

Proof. Assume that $F(\tau)=\max_{t\in[0,1]}F(t)=\|F\|_1$. For $t\in[0,1]$, we have

$$\frac{G(t,s)}{G(\tau,s)}=\begin{cases}\frac{t(1-s)}{(1-\tau)s}=t(1-t)\frac{(1-s)}{(1-\tau)}\frac{1}{(1-t)s}\ge t(1-t), & t\le s\le\tau\\ \frac{(1-t)s}{\tau(1-s)}=t(1-t)\frac{s}{\tau}\frac{1}{t(1-s)}\ge t(1-t), & \tau\le s\le t\\ \frac{t(1-s)}{\tau(1-s)}=t(1-t)\frac{1}{\tau(1-t)}\ge t(1-t), & t,\tau\le s\\ \frac{(1-t)s}{(1-\tau)s}=t(1-t)\frac{1}{t(1-\tau)}\ge t(1-t), & t,\tau\ge s\end{cases}=q(t),\quad t\in[0,1].$$

Then, for $t\in[0,1]$, we have

$$\begin{aligned}F(t)&=\int_0^1 G(t,s)\overline{\Phi}(s)\,ds=\int_0^1\frac{G(t,s)}{G(\tau,s)}G(\tau,s)\overline{\Phi}(s)\,ds\\ &\ge t(1-t)\int_0^1 G(\tau,s)\overline{\Phi}(s)\,ds=t(1-t)F(\tau)=t(1-t)\|F\|_1,\end{aligned}$$

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which implies that

$$F\left(\frac{1}{2}\right) \geq \frac{1}{2}\left(1-\frac{1}{2}\right)\|F\|_1 = \frac{1}{4}\|F\|_1, \quad \text{i.e.,} \quad \|F\|_1 \leq 4F\left(\frac{1}{2}\right).$$

Moreover, since $F'(t) = -\int_0^t s\overline{\Phi}(s)\,ds + \int_t^1 (1-s)\overline{\Phi}(s)\,ds$ for all $t \in [0,1]$ and $F(t) \geq 0$, one has

$$\begin{aligned} t(1-t)|F'(t)| &\leq t(1-t)\int_0^t s\overline{\Phi}(s)\,ds + (1-t)t\int_t^1 (1-s)\overline{\Phi}(s)\,ds \\ &\leq (1-t)\int_0^t s\overline{\Phi}(s)\,ds + t\int_t^1 (1-s)\overline{\Phi}(s)\,ds \\ &= \int_0^1 G(t,s)\overline{\Phi}(s)\,ds \leq \|F\|_1 \leq 4F\left(\tfrac{1}{2}\right), \quad t \in [0,1], \end{aligned} \tag{2.8}$$

which means that $\|F\|_2 = \sup_{t\in(0,1)} t(1-t)|F'(t)| \leq \|F\|_1 \leq 4F(\frac{1}{2})$. Consequently, $F \in P$. □

Now we list some conditions which will be needed in Sections 3 and 4:

$$\Phi \in C[0,1] \text{ with } \Phi(t) > 0 \text{ on } (0,1), \tag{2.9}$$

and

$$\begin{cases} f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty) \text{ is continuous with} \\ f(t,x,y) > 0 \text{ for } (t,x,y) \in [0,1] \times [0,+\infty) \times [0,+\infty) \\ f(t,x,y) \leq [h(x)+w(x)][g(y)+r(y)] \\ \text{on } [0,1] \times [0,+\infty) \times [0,+\infty) \text{ with } w > 0, g > 0 \\ \text{continuous and nonincreasing on } [0,+\infty), \\ \int_0^1 \Phi(s) r\left(k_0 \frac{1}{s(1-s)}\right) ds < +\infty, \quad \text{for all } k_0 > 0, \text{ and} \\ h \geq 0, \quad r \geq 0 \text{ continuous and nondecreasing on } [0,\infty). \end{cases} \tag{2.10}$$

For $x \in P$, define an operator by

$$(Ax)(t) = \int_0^1 G(t,s)\Phi(s) f(s, x(s), |x'(s)|)\,ds, \quad t \in [0,1]. \tag{2.11}$$

LEMMA 2.6. *Assume that* (2.9) *and* (2.10) *hold. Then $A : P \to P$ is continuous and completely continuous. Moreover, for every $x \in P$, $Ax \in C^1[0,1]$.*

Proof. First, we show that $A : P \to P$ is well defined and $Ax \in C^1[0,1]$ for each $x \in P$. For $x \in P$, Lemma 2.1 yields $|x'(t)| \leq \frac{\|x\|}{t(1-t)}$, $t \in (0,1)$. Then, from (2.9) and (2.10), we have that for $t \in [0,1]$,

$$\begin{aligned} |(Ax)(t)| &= \left| \int_0^1 G(t,s)\Phi(s) f(s, x(s), |x'(s)|)\,ds \right| \\ &\leq \int_0^1 G(t,s)\Phi(s)[h(x(s)) + w(x(s))][g(|x'(s)|) + r(|x'(s)|)]\,ds \\ &\leq \int_0^1 G(t,s)\Phi(s)[h(\|x\|) + w(0)]\left[g(0) + r\left(\frac{\|x\|}{s(1-s)}\right)\right] ds < +\infty \end{aligned}$$

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and

$$\begin{aligned}|(Ax)'(t)| &= \left| -\int_0^t s\Phi(s) f(s, x(s), |x'(s)|)\, ds + \int_t^1 (1-s)\Phi(s) f(s, x(s), |x'(s)|)\, ds \right| \\ &\le \int_0^1 \Phi(s)|f(s, x(s), |x'(s)|)|\, ds \\ &\le \int_0^1 \Phi(s)[h(\|x\|) + w(0)] \left[g(0) + r\left(\frac{\|x\|}{s(1-s)} \right) \right] ds < +\infty, \quad t \in [0, 1],\end{aligned}$$

which means that A is well defined. From Lemma 2.5, for every $x \in P$, we have $Ax \in P$, which implies that $AP \subseteq P$. Moreover, since

$$\begin{aligned}&|(Ax)'(t_1) - (Ax)'(t_2)| \\ &\le \left| \int_{t_1}^1 (1-s)\Phi(s) f(s, x(s), |x'(s)|)\, ds - \int_{t_2}^1 (1-s)\Phi(s) f(s, x(s), |x'(s)|)\, ds \right| \\ &\quad + \left| -\int_0^{t_1} s\Phi(s) f(s, x(s), |x'(s)|)\, ds + \int_0^{t_2} s\Phi(s) f(s, x(s), |x'(s)|)\, ds \right| \\ &\le 2 \left| \int_{t_1}^{t_2} \Phi(s)[h(\|x\|) + w(0)] \left[g(0) + r\left(\frac{\|x\|}{s(1-s)} \right) \right] ds \right|,\end{aligned}$$

we see that $\lim_{t\to 0+}(Ax)'(t)$ exists and $\lim_{t\to 1-}(Ax)'(t)$ exists, which means that $Ax \in C^1[0, 1]$.

Now we show that $A : P \to P$ is continuous. Assume that $\{x_m\}_{m=1}^{\infty} \subseteq P$ and $x_0 \in P$ with $\lim_{m\to+\infty} x_m = x_0$. Then, there exists an $M > 0$ such that $\|x_m\| < M$ for all $m \in \{1, 2, \dots\}$ (note that Lemma 2.1 guarantees that $|x_m'(t)| \le \frac{M}{t(1-t)}$, $\forall t \in (0, 1)$). Thus, $\lim_{m\to+\infty} f(t, x_m(t), |x_m'(t)|) = f(t, x_0(t), |x_0'(t)|)$, $\quad t \in (0, 1)$ and

$$|f(t, x_m(t), |x_m'(t)|)| \le [h(M) + w(0)] \left[g(0) + r\left(\frac{M}{t(1-t)} \right) \right].$$

The Lebesgue Dominated Convergence Theorem guarantees that

$$\begin{aligned}&\|Ax_m - Ax_0\|_1 \\ &= \max_{t\in[0,1]} \left| \int_0^1 G(t, s)\Phi(s)[f(s, x_m(s), |x_m'(s)|) - f(s, x_0(s), |x_0'(s)|)]\, ds \right| \\ &\le \int_0^1 \Phi(s)|f(s, x_m(s), |x_m'(s)|) - f(s, x_0(s), |x_0'(s)|)|\, ds \to 0, \quad \text{as } m \to +\infty\end{aligned}$$

and

$$\begin{aligned}&\|Ax_m - Ax_0\|_2 \\ &= \sup_{t\in(0,1)} t(1-t) \Big| -\int_0^t s\Phi(s)[f(s, x_m(s), |x_m'(s)|) - f(s, x_0(s), |x_0'(s)|)]\, ds \\ &\quad + \int_t^1 (1-s)\Phi(s)[f(s, x_m(s), |x_m'(s)|) - f(s, x_0(s), |x_0'(s)|)]\, ds \Big| \\ &\le \int_0^1 \Phi(s)|f(s, x_m(s), |x_m'(s)|) - f(s, x_0(s), |x_0'(s)|)|\, ds \to 0, \quad \text{as } m \to +\infty,\end{aligned}$$

which implies that $\lim_{m\to+\infty} \|Ax_m - Ax_0\| = 0$. Hence, $A : P \to P$ is continuous.

Finally we show for any bounded $D \subseteq P$, $A(D)$ is relatively compact. Since D is bounded, there exists an $M > 0$ such that $\|x\| \leq M$ for all $x \in D$ (note that Lemma 2.1 guarantees that $|x'(t)| \leq \frac{M}{t(1-t)}$, $\forall t \in (0, 1)$). Now (2.9) and (2.10) yield

$$\begin{aligned}
\|Ax\|_1 &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\Phi(s) f(s, x(s), |x'(s)|)\, ds \right| \\
&\leq \int_0^1 \Phi(s)[h(x(s)) + w(x(s))][g(|x'(s)|) + r(|x'(s)|)]\, ds \\
&\leq \int_0^1 \Phi(s)[h(M) + w(0)] \left[g(0) + r\left(\frac{M}{s(1-s)}\right)\right] ds
\end{aligned}$$

which implies that the functions belonging to $\{(AD)(t)\}$ are uniformly bounded on $[0, 1]$, and

$$\begin{aligned}
&\sup_{t\in(0,1)} |(Ax)'(t)| \\
&= \sup_{t\in[0,1]} \left| -\int_0^t s\Phi(s) f(s, x(s), |x'(s)|)\, ds + \int_t^1 (1-s)\Phi(s) f(s, x(s), |x'(s)|)\, ds \right| \\
&\leq \int_0^1 \Phi(s)[h(x(s)) + w(x(s))][g(|x'(s)|) + r(|x'(s)|)]\, ds \\
&\leq \int_0^1 \Phi(s)[h(M) + w(0)] \left[g(0) + r\left(\frac{M}{s(1-s)}\right)\right] ds.
\end{aligned}$$

which implies that the functions belonging to $\{(AD)'(t)\}$ are uniformly bounded on $[0, 1]$ and the functions belonging to $\{(AD)(t)\}$ are equicontinuous on $[0, 1]$.

For any $t_1, t_2 \in [0, 1]$, $x \in D$, we have

$$\begin{aligned}
&|(Ax)'(t_1) - (Ax)'(t_2)| \\
&= \left| -\int_0^{t_1} s\Phi(s) f(s, x(s), |x'(s)|)\, ds + \int_0^{t_2} s\Phi(s) f(s, x(s), |x'(s)|)\, ds \right. \\
&\quad \left. + \int_{t_1}^1 (1-s)\Phi(s) f(s, x(s), |x'(s)|)\, ds - \int_{t_2}^1 (1-s)\Phi(s) f(s, x(s), |x'(s)|)\, ds \right| \\
&\leq 2 \left| \int_{t_1}^{t_2} \Phi(s)[h(M) + w(0)] \left[g(0) + r\left(\frac{M}{s(1-s)}\right)\right] ds \right|.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(Ax)'(t_1) - (Ax)'(t_2)| < \varepsilon$, $\forall |t_1 - t_2| < \delta$, $x \in D$, which means that the functions from $\{(Ax)'(t), x \in D\}$ are equicontinuous on $[0, 1]$.

By the Arzela-Ascoli theorem, $A(D)$ is relatively compact under the ordinary norm $\|Ax\|_0 = \max\{\max\{|(Ax)(t)|, t \in [0, 1]\}, \max\{|(Ax)'(t)|, t \in [0, 1]\}\}$. Since the new norm $\|Ax\|$ is less than the ordinary norm $\|Ax\|_0$, we know that AD is relatively compact under the new norm $\|\cdot\|$.

Hence, $A : P \to P$ is continuous and completely continuous. □

3. Multiple positive solutions to (1.1) with singularity at $x' = 0$ but not at $x = 0$. In this section our nonlinearity f may be singular at $x' = 0$ but not at $x = 0$. Throughout this section we will assume that the following conditions hold:

$$\Phi \in C[0,1] \text{ with } \Phi(t) > 0 \text{ on } (0,1), \tag{3.1}$$

$$\begin{cases} f:[0,1]\times[0,+\infty)\times(0,+\infty)\to(0,+\infty) \text{ is continuous with} \\ f(t,x,y) > 0 \text{ for } (t,x,y)\in[0,1]\times[0,+\infty)\times(0,+\infty), \\ f(t,x,y) \le h(x)[g(y)+r(y)] \text{ on } [0,1]\times[0,+\infty)\times(0,+\infty) \text{ with} \\ \int_0^1 \Phi(s) r\big(k_0 \frac{1}{s(1-s)}\big)\,ds < +\infty, \quad \text{for all } k_0 > 0, \text{ and} \\ h \ge 0, \ \ r \ge 0 \text{ continuous and nondecreasing on } [0,\infty) \\ g > 0 \text{ continuous and nonincreasing on } (0,\infty), \end{cases} \tag{3.2}$$

$$\begin{cases} \sup_{c\in(0,+\infty)} \frac{c}{I^{-1}(h(c)\int_0^1 \Phi(s)\,ds)} > 1, \\ \text{where } \ I(z) = \int_0^z \frac{1}{g(u)+r(u)}\,du, \ z\in(0,+\infty), \end{cases} \tag{3.3}$$

$$\begin{cases} \text{for constants } H, H' > 0 \text{ there exists a function } \psi_{H,H'} \\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \\ \text{and a positive constant } 0 \le \gamma < 1 \text{ such that} \\ f(t,x,y) \ge \psi_H(t) x^{\gamma} \text{ on } [0,1]\times[0,H]\times(0,H'], \end{cases} \tag{3.4}$$

and

$$\begin{cases} \text{there exists a } g_1 \in C([0,+\infty)\times(0,+\infty),(0,+\infty)) \text{ with} \\ f(t,x,y) \ge g_1(x,y), \forall (t,x,y)\in[0,1]\times[0,+\infty)\times(0,+\infty) \text{ such that} \\ \lim_{x\to+\infty} \frac{g_1(x,y)}{x} = +\infty \text{ uniformly for } y\in(0,+\infty). \end{cases} \tag{3.5}$$

THEOREM 3.1. *Suppose that* (3.1)–(3.5) *hold. Then* (1.1) *has at least two nonnegative solutions* $x_{0,1}, x_{0,2} \in C^1[0,1]\cap C^2(0,1)$ *with* $x_{0,1}(t) > 0$ *and* $x_{0,2}(t) > 0$ *on* $(0,1)$.

Proof. From (3.3) and the continuity of I^{-1} and h, choose an $R_1 > 0$, and a $\varepsilon > 0$ with $\varepsilon < \frac{R_1}{2}$ and

$$\frac{R_1}{I^{-1}(I(\varepsilon) + h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds)} > 1. \tag{3.6}$$

Let $n_0 \in \{1,2,\ldots\}$ be chosen so that $\frac{1}{n_0} < \varepsilon$, and let $N_0 = \{n_0, n_0+1, \ldots\}$. For each $n \in N_0$, for $x \in P$, define

$$(A_n x)(t) = \int_0^1 G(t,s)\Phi(s) f\left(s, x(s) + \frac{1}{n}s, |x'(s)| + \frac{1}{n}\right) ds, \quad t\in[0,1]. \tag{3.7}$$

It is easy to see that $f(t, x+\frac{1}{n}t, |x'|+\frac{1}{n}) \le h(x+\frac{1}{n})[g(\frac{1}{n}) + r(|x'|+\frac{1}{n})]$, $(t,x,x') \in [0,1]\times[0,+\infty)\times[0,+\infty)$. Now (3.2) implies that (2.10) is true for each $n\in N_0$. Then, Lemma 2.6 guarantees that $A_n : P \to P$ is continuous and completely continuous with $A_n P \subseteq C^1[0,1]$ for each $n \in N_0$.

Let

$$\Omega_1 = \big\{x \in C_q^1[0,1] : \|x\| < R_1\big\}.$$

Choose $0 < a^* < b^* < 1$ and let

$$N^* = \left(a^*(1-b^*)\frac{1}{4}\min_{t\in[a^*,b^*]}\int_{a^*}^{b^*} G(t,s)\Phi(s)\,ds\right)^{-1} + 1. \tag{3.8}$$

Now from (3.5), there exists an $R_2 > R_1$ such that

$$g_1(x,y) \geq N^* x, \quad \forall x \geq R_2,\, y \in (0,+\infty). \tag{3.9}$$

Let

$$\Omega_2 = \left\{x \in C_q^1[0,1] : \|x\| < \frac{4R_2}{a^*(1-b^*)}\right\}.$$

Then, for each $n \in N_0$, we claim that

$$\mu A_n x \neq x, \quad \forall \mu \in (0,1], \quad x \in P \cap \partial\Omega_1, \tag{3.10}$$

and

$$A_n x \not\leq x, \quad \forall x \in P \cap \partial\Omega_2. \tag{3.11}$$

First we show that (3.10) is true. Suppose there exists an $x_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $x_0 = \mu_0 A_n x_0$. Then

$$x_0''(t) + \mu_0\Phi(t)f\left(t, x_0(t) + \frac{1}{n}t, |x_0'(t)| + \frac{1}{n}\right) = 0,\ t \in (0,1),\ \text{and}\ x_0(0) = 0,\ x_0(1) = 0,$$

which means that $x_0(t) > 0$ on $(0,1)$ with $x_0(0) = x_0(1) = 0$ and $x_0'(t)$ is decreasing on $(0,1)$. Thus, there exists a unique $t_0 \in (0,1)$ with $x_0'(t_0) = 0$, $x_0'(t) > 0$ on $(0,t_0)$ and $x_0'(t) < 0$ on $(t_0,1)$ and $\|x_0\|_1 = x_0(t_0)$. From (3.2), we have

$$-x_0''(t) \leq \Phi(t)h\left(x_0(t) + \frac{1}{n}t\right)\left[g\left(x_0'(t) + \frac{1}{n}\right) + r\left(x_0'(t) + \frac{1}{n}\right)\right], \forall t \in (0,t_0),$$

which means that

$$\frac{-x_0''(t)}{g\left(x_0'(t) + \frac{1}{n}\right) + r\left(x_0'(t) + \frac{1}{n}\right)} \leq h\left(x_0(t) + \frac{1}{n}t\right)\Phi(t), \quad \forall t \in (0,t_0). \tag{3.12}$$

Integration from t to t_0 yields $I(x_0'(t) + \frac{1}{n}) - I(x_0'(t_0) + \frac{1}{n}) = I(x_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) \leq h(x_0(t_0) + \frac{1}{n})\int_0^1 \Phi(s)\,ds$, i.e., $I(x_0'(t) + \frac{1}{n}) < I(\varepsilon) + h(R_1 + \varepsilon)\int_0^1 \Phi(s)\,ds$. Then,

$$x_0'(t) < x_0'(t) + \frac{1}{n} < I^{-1}\left(I(\varepsilon) + h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds\right), \forall t \in (0,t_0]. \tag{3.13}$$

Now integrate from 0 to t_0 to obtain

$$\|x_0\|_1 = x_0(t_0) = x_0(t_0) - x_0(0) \leq I^{-1}\left(I(\varepsilon) + h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds\right). \tag{3.14}$$

Similarly, if $t > t_0$, we have

$$\frac{-x_0''(t)}{g\big(-x_0'(t)+\frac{1}{n}\big)+r\big(-x_0'(t)+\frac{1}{n}\big)} \le h\left(x_0(t)+\frac{1}{n}t\right)\Phi(t), \tag{3.15}$$

and $I(-x_0'(t)+\frac{1}{n}) - I(-x_0'(t_0)+\frac{1}{n}) = I(-x_0'(t)+\frac{1}{n}) - I(\frac{1}{n}) \le \int_0^1 \Phi(s)\,ds\, h(R_1+\varepsilon)$, which implies

$$-x_0'(t) \le I^{-1}\left(I(\varepsilon)+h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds\right), \quad t \in (t_0, 1). \tag{3.16}$$

Now (3.13) and (3.16) guarantee that

$$\|x_0\|_2 = \sup_{t\in(0,1)} t(1-t)|x'(t)| \le I^{-1}\left(I(\varepsilon)+h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds\right). \tag{3.17}$$

Combining with (3.14) one has $R_1 = \max\{\|x_0\|_1, \|x_0\|_2\} \le I^{-1}(I(\varepsilon)+h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds)$, which means

$$\frac{R_1}{I^{-1}\big(I(\varepsilon)+h(R_1+\varepsilon)\int_0^1 \Phi(s)\,ds\big)} \le 1,$$

a contradiction to (3.6). Then, (3.10) is true.

Next we show that (3.11) is true. Suppose that there is an $x_0 \in P\cap\partial\Omega_2$ with $x_0 \ge A_n x_0$. By the definition of the cone and Lemma 2.4, one has $x_0(t) \ge t(1-t)\|x_0\|_1 \ge a^*(1-b^*)\frac{1}{4}\|x_0\| = a^*(1-b^*)\frac{1}{4}\frac{4R_2}{a^*(1-b^*)} = R_2, \quad \forall t \in [a^*, b^*]$. Clearly for $t \in [a^*, b^*]$, $x_0(t)+\frac{1}{n}t \ge R_2, \quad \forall t \in [a^*, b^*]$. Then, for $t \in [a^*, b^*]$, from (3.9), one has

$$\begin{aligned}
x_0(t) \ge (A_n x_0)(t) &\ge \int_{a^*}^{b^*} G(t,s)\Phi(s) g_1\left(x_0(s)+\frac{1}{n}s, |x_0'(s)|+\frac{1}{n}\right) ds \\
&\ge \int_{a^*}^{b^*} G(t,s)\Phi(s) N^*\left(x_0(s)+\frac{1}{n}s\right) ds \ge \int_{a^*}^{b^*} G(t,s)\Phi(s)\,ds N^* R_2 \\
&\ge a^*(1-b^*)\frac{1}{4}\min_{t\in[a^*,b^*]}\int_{a^*}^{b^*} G(t,s)\Phi(s)\,ds N^*\frac{4R_2}{a^*(1-b^*)} > \frac{4R_2}{a^*(1-b^*)},
\end{aligned}$$

which implies that $\|x_0\| \ge \|x_0\|_1 > \frac{4R_2}{a^*(1-b^*)}$, a contradiction to $x_0 \in P\cap\partial\Omega_2$. Thus, (3.11) is true.

From Lemma 2.2 and Lemma 2.3, for each $n \in N_0$, we have

$$i(A_n, P\cap\Omega_1, P) = 1, \text{ and } i(A_n, P\cap\Omega_2, P) = 0,$$

which implies that

$$i(A_n, P\cap(\Omega_2-\overline{\Omega}_1), P) = -1, \quad n \in N_0. \tag{3.18}$$

As a result, for each $n \in N_0$, there exist $x_{n,1} \in P\cap\Omega_1$ and $x_{n,2} \in P\cap(\Omega_2-\overline{\Omega}_1)$ such that $x_{n,1} = A_n x_{n,1}$ and $x_{n,2} = A_n x_{n,2}$.

Now we consider $\{x_{n,1}\}_{n\in N_0}$ and $\{x_{n,2}\}_{n\in N_0}$. First we show that $\{x_{n,1}\}$ is relatively compact in $C^1[0, 1]$. Clearly the functions belonging to $\{x_{n,1}(t)\}$ are uniformly bounded with $\max_{t\in[0,1]} |x_{n,1}(t)| \le R_1$, $n \in N_0$.

For each $n \in N_0$, one has

$$x''_{n,1}(t) + \Phi(t)f\left(t, x_{n,1}(t) + \frac{1}{n}t, |x'_{n,1}(t)| + \frac{1}{n}\right) = 0,\ t \in (0,1),\ \text{ and }$$
$$x_{n,1}(0) = 0,\ x_{n,1}(1) = 0,$$

which means that $x_{n,1}(t) > 0$ on $(0, 1)$ with $x_{n,1}(0) = x_{n,1}(1) = 0$ and $x'_{n,1}(t)$ is decreasing on $(0, 1)$. Thus, there exists a unique $t_n \in (0, 1)$ with $x'_{n,1}(t_n) = 0$, $x'_{n,1}(t_n) > 0$ on $(0, t_n)$ and $x'_{n,1}(t) < 0$ on $(t_n, 1)$ and $\|x_{n,1}\|_1 = x_{n,1}(t_n)$. Following the argument used to prove (3.12) and (3.15) yields

$$\frac{-x''_{n,1}(t)}{g\big(|x'_{n,1}(t)| + \frac{1}{n}\big) + r\big(|x'_{n,1}(t)| + \frac{1}{n}\big)} \le h\left(x_{n,1}(t) + \frac{1}{n}t\right)\Phi(t),\quad \forall t \in (0, t_n) \cup (t_n, 1). \tag{3.19}$$

A similar argument to that used to prove (3.13) and (3.16) yields

$$x'_{n,1}(t) < x'_{n,1}(t) + \frac{1}{n} < I^{-1}\left(I(\varepsilon) + h(R_1 + \varepsilon)\int_0^1 \Phi(s)\,ds\right), \forall t \in (0, t_n], \tag{3.20}$$

and

$$-x'_{n,1}(t) < I^{-1}\left(I(\varepsilon) + h(R_1 + \varepsilon)\int_0^1 \Phi(s)\,ds\right),\quad t \in (t_n, 1). \tag{3.21}$$

Now (3.20) and (3.21) guarantee that

$$\sup_{n\ge n_0}\ \sup_{t\in(0,1)} |x'_{n,1}(t)| \le I^{-1}\left(I(\varepsilon) + h(R_1 + \varepsilon)\int_0^1 \Phi(s)\,ds\right). \tag{3.22}$$

Consequently, the functions belonging to $\{x'_{n,1}(t)\}$ are uniformly bounded on $[0, 1]$, which implies that the functions belonging to $\{x_{n,1}(t)\}$ are equicontinuous on $[0, 1]$.

Now (3.19) yields

$$\begin{aligned}\left|I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| &\le \left|\int_{t_1}^{t_2} \|\Phi\|_1 h(R_1 + \varepsilon)\,ds\right| \\ &= \|\Phi\|_1 h(R_1 + \varepsilon)|t_1 - t_2|,\end{aligned} \tag{3.23}$$

if $t_2 > t_1 > t_n$ and

$$\begin{aligned}\left|I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| &\le \left|\int_{t_1}^{t_2} \|\Phi\|_1 h(R_1 + \varepsilon)\,ds\right| \\ &= \|\Phi\|_1 h(R_1 + \varepsilon)|t_1 - t_2|,\end{aligned} \tag{3.24}$$

if $t_2 < t_1 < t_n$.

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From (3.23) and (3.24), one has

$$\begin{cases} \left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)\right| \\ \quad \le \left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(x'_{n,1}(t_n)+\frac{1}{n}\right)\right|+\left|I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)-I\left(x'_{n,1}(t_n)+\frac{1}{n}\right)\right| \\ \quad \le \|\Phi\|_1 h(R_1+\varepsilon)(|t_1-t_n|+|t_2-t_n|)=\|\Phi\|_1 h(R_1+\varepsilon)|t_1-t_2|, \\ \text{if } t_2>t_n>t_1 \text{ or} \\ \left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)\right| \\ \quad \le \left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(x'_{n,1}(t_n)+\frac{1}{n}\right)\right|+\left|I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)-I\left(x'_{n,1}(t_n)+\frac{1}{n}\right)\right| \\ \quad \le \|\Phi\|_1 h(R_1+\varepsilon)(|t_1-t_n|+|t_2-t_n|)=\|\Phi\|_1 h(R_1+\varepsilon)|t_1-t_2|, \\ \text{if } t_2<t_n<t_1. \end{cases} \tag{3.25}$$

Thus, for any $\varepsilon'>0$, there is a $\delta'>0$ such that

$$\left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)\right|<\varepsilon',\quad \forall|t_2-t_1|<\delta',\ t_1,t_2\in[0,1],$$

which means that the functions belonging to $\{I(|x'_{n,1}(t)|+\frac{1}{n})\}$ are equicontinuous on $[0,1]$. By the continuity of $I^{-1}(z)$, we know that the functions belonging to $\{|x'_{n,1}(t)|+\frac{1}{n}\}$ are equicontinuous on $[0,1]$, which implies the functions belonging to $\{|x'_{n,1}(t)|\}$ are equicontinuous on $[0,1]$. Then, for any $\varepsilon'>0$, there is a $\delta>0$ such that $\||x'_{n,1}(t_1)|-|x'_{n,1}(t_2)|\|<\varepsilon', \forall|t_1-t_2|<\delta, t_1,t_2\in[0,1]$, which implies $|x'_{n,1}(t_1)-x'_{n,1}(t_2)|=\||x'_{n,1}(t_1)|-|x'_{n,1}(t_2)|\|<\varepsilon', \forall|t_1-t_2|<\delta, t_1,t_2\le t_n\in[0,1]$, $|x'_{n,1}(t_1)-x'_{n,1}(t_2)|=\||x'_{n,1}(t_1)|-|x'_{n,1}(t_2)|\|<\varepsilon', \forall|t_1-t_2|<\delta, t_1,t_2\ge t_n\in[0,1]$ and (notice $x'_{n,1}(t_n)=0$)

$$\begin{aligned} |x'_{n,1}(t_1)-x'_{n,1}(t_2)| &\le |x'_{n,1}(t_1)-x'_{n,1}(t_n)|+|x'_{n,1}(t_n)-x'_{n,1}(t_2)| \\ &= \||x'_{n,1}(t_1)|-|x'_{n,1}(t_n)|\|+\||x'_{n,1}(t_n)|-|x'_{n,1}(t_2)|\| \\ &< 2\varepsilon', \forall|t_1-t_2|<\delta,\quad t_1<t_n<t_2\in[0,1],\ \text{or}\ \ t_2<t_n<t_1\in[0,1], \end{aligned}$$

which means that the functions belonging to $\{x'_{n,1}(t)\}$ are equicontinuous on $[0,1]$.

Consequently, the Arzela-Ascoli Theorem guarantees that $\{x_{n,1}\}$ is relatively compact in $C^1[0,1]$, which means that there is a convergent subsequence $\{x_{n_j,1}\}$ of $\{x_{n,1}\}$ such that $\lim_{j\to+\infty} x_{n_j,1}=x_{0,1}\in C^1[0,1]$ (under the ordinary norm $\|x\|_0=\max\{\max_{t\in[0,1]}|x(t)|,\max_{t\in[0,1]}|x'(t)|\}$). Also recall that $x'_{n_j,1}(t_{n_j})=0$ and $x'_{n_j,1}(t)\ne 0$ if $t\ne t_{n_j}$. Without loss of generality, we assume that $t_{n_j}\to t_0\in[0,1]$. Clearly $x'_{0,1}(t_0)=0$ (keep also in mind that $\{x'_{n_j,1}\}$ converges to $x'_{0,1}$ uniformly on $[0,1]$ and $x'_{0,1}$ is continuous on $[0,1]$).

We now show that

$$t_0\in(0,1)\ \text{ and }\ x'_{0,1}(t)\ne 0\ \text{ if }\ t\ne t_0. \tag{3.26}$$

Suppose that $t_0=0$. Let $R'=I^{-1}(I(\varepsilon)+h(R_1+\varepsilon)\int_0^1\Phi(s)\,ds)+1$. >From (3.4), there is a $\Psi_{R_1+\varepsilon,R'}\in C[0,1]$ with $\Psi_{R_1+\varepsilon,R'}(t)>0$ for all $t\in(0,1)$ such that $f(t,x_{n,1}(t)+\frac{1}{n}t,|x'_{n,1}(t)|+\frac{1}{n})\ge\Psi_{R_1+\varepsilon,R'}(t)(x_{n,1}(t)+\frac{1}{n}t)^\gamma\ge\Psi_{R_1+\varepsilon,R'}(t)(x_{n,1}(t))^\gamma$, $\quad t\in[0,1]$. Thus (note $x_{n,1} \in P$ and Lemma 2.5),

$$\begin{aligned} x_{n,1}(t) &\geq \int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(x_{n,1}(s))^\gamma\, ds \\ &\geq \int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma\, ds\|x_{n,1}\|_1^\gamma \\ &\geq t(1-t)\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma\, ds\|x_{n,1}\|_1^\gamma, \quad \forall t\in[0,1], \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n,1}\|_1 &\geq \left(\max_{t\in[0,1]} t(1-t)\right)\left(\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma\, ds\|x_{n,1}\|_1^\gamma\right) \\ &= \frac{1}{4}\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma\, ds\|x_{n,1}\|_1^\gamma \end{aligned}$$

and so $\|x_{n,1}\|_1 \geq (\frac{1}{4}\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma\, ds)^{\frac{1}{1-\gamma}} \overset{def.}{=} a_0 > 0$. Consequently,

$$x_{n,1}(t) \geq t(1-t)\|x_{n,1}\|_1 \geq t(1-t)a_0, \quad t\in[0,1]. \tag{3.27}$$

Now (3.27) implies that for each $n \in N_0$,

$$-x''_{n,1}(t) \geq \Phi(t)\Psi_{R_1+\varepsilon,R'}(t)(t(1-t))^\gamma a_0^\gamma, \quad t\in(0,1), \tag{3.28}$$

and so $x'_{n_j,1}(t_{n_j}) - x'_{n_j,1}(t) \geq \int_{t_{n_j}}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds$, $t > t_{n_j}$. Letting $j \to +\infty$, one has $x'_{0,1}(t_0) - x'_{n_j,1}(t) \geq \int_{t_0}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds$, $t > t_0 = 0$, i.e., $x'_{n_j,1}(t) \leq -\int_0^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds < 0$, $t > 0$, which means $x_{0,1}(t)$ is decreasing on $[0,1]$, a contradiction to $x_{0,1}(0) = x_{0,1}(1)$. Thus, $t_0 > 0$. Similarly, we get $t_0 < 1$. Hence, $t_0 \in (0,1)$.

From (3.28), we have

$$\begin{aligned} x'_{n_j,1}(t_{n_j}) - x'_{n_j,1}(t) &\geq \int_{t_{n_j}}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds, \quad t > t_{n_j}, \\ x'_{n_j,1}(t) - x'_{n_j,1}(t_{n_j}) &\geq \int_t^{t_{n_j}} \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds, \quad t < t_{n_j}. \end{aligned}$$

Letting $j \to +\infty$, one has

$$\begin{aligned} -x'_{0,1}(t) = x'_{0,1}(t_0) - x'_{0,1}(t) &\geq \int_{t_0}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds, \quad t > t_0, \\ x'_{0,1}(t) = x'_{0,1}(t) - x'_{0,1}(t_0) &\geq \int_t^{t_0} \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)(s(1-s))^\gamma a_0^\gamma\, ds, \quad t < t_0. \end{aligned}$$

Consequently, one has $x'_{0,1}(t) \neq 0$, if $t \neq t_0$. Therefore, (3.26) is true, which implies that, for any $\frac{\min\{t_0, 1-t_0\}}{2} > \varepsilon' > 0$, there exists an $N > 0$ such that $t_{n_j} \in [t_0 - \frac{\varepsilon'}{2}, t_0 + \frac{\varepsilon'}{2}]$

(notice that $x'_{n_j,1}(t_{n_j}) = 0$) for all $j \geq N$. Of course

$$\min\left\{\inf_{j\geq N}\inf_{t\in[0,t_0-\varepsilon']}|x'_{n_j}(t)|, \inf_{j\geq N}\inf_{t\in[t_0+\varepsilon',1]}|x'_{n_j}(t)|\right\} = \delta_0 > 0,$$

and so,

$$\begin{aligned}\Phi(t)&\left|f\left(t, x_{n_j,1}(t) + \frac{1}{n_j}t, |x'_{n_j,1}(t)| + \frac{1}{n_j}\right)\right| \\ &\leq h(R_1+\varepsilon)[g(\delta_0) + r(R')]\Phi(t), \quad t \in (0, t_0-\varepsilon'] \cup [t_0+\varepsilon', 1).\end{aligned}$$

The Lebesgue Dominated Convergence Theorem guarantees that

$$\begin{aligned}x'_{0,1}(t) - x'_{0,1}(t_0+\varepsilon') &= \lim_{j\to+\infty}[x'_{n_j,1}(t) - x'_{n_j,1}(t_0+\varepsilon')] \\ &= \lim_{j\to+\infty}\int_{t_0+\varepsilon'}^{t}\Phi(s)f\left(s, x_{n_j,1}(s) + \frac{1}{n_j}s, |x'_{n_j,1}(s)| + \frac{1}{n_j}\right)ds \\ &= \int_{t_0+\varepsilon'}^{t}\Phi(s)f(s, x_{0,1}(s), |x'_{0,1}(s)|)\,ds, \quad t \in [t_0+\varepsilon', 1)\end{aligned}$$

and similarly

$$x'_{0,1}(t) - x'_{0,1}(t_0-\varepsilon') = \int_{t_0-\varepsilon'}^{t}\Phi(s)f(s, x_{0,1}(s), |x'_{0,1}(s)|)\,ds, \quad t \in (0, t_0-\varepsilon'],$$

which implies that $-x''_{0,1}(t) = \Phi(t)f(t, x_{0,1}(t), |x'_{0,1}(t)|), \quad t \in (0, t_0-\varepsilon'] \cup [t_0+\varepsilon', 1)$. Since ε' is arbitrary, we have $-x''_{0,1}(t) = \Phi(t)f(t, x_{0,1}(t), |x'_{0,1}(t)|), \quad t \in (0, t_0) \cup (t_0, 1)$. In addition $x_{0,1}(0) = x_{0,1}(1) = 0$, and $x_{0,1}$ is a solution of (1.1) with $x_{0,1}(t) > 0$ on $(0, 1)$ and $\|x_{0,1}\| < R_1$.

For the set $\{x_{n,2}\}_{n\in N_0} \subseteq (\Omega_2 - \overline{\Omega}_1) \cap P$, a similar proof yields a convergent subsequence $\{x_{n_i,2}\}$ of $\{x_{n,2}\}$ with $\lim_{i\to+\infty} x_{n_i,2} = x_{0,2} \in C^1[0, 1]$. Also $x_{0,2}$ is a nonnegative solution of (1.1) with $x_{0,2}(t) > 0$ on $(0, 1)$ and $R_1 < \|x_{0,2}\| < \frac{4R_2}{a^*(1-b^*)}$.

Consequently, (1.1) has at least two different nonnegative solutions $x_{0,1}$ and $x_{0,2}$ with $\|x_{0,1}\| < R_1 < \|x_{0,2}\|$. □

EXAMPLE 3.1. Consider the boundary value problems

$$\begin{cases} x'' + \mu(y^b + y^d)(1 + |x'|^{-a}) = 0 \\ x(0) = 0, \; x(1) = 0, \end{cases} \tag{3.29}$$

with $0 \leq a$, $b > 1$, $1 > d \geq 0$ and $\mu > 0$. If

$$\mu < \frac{1}{1+a}\left(\sup_{c\in(0,+\infty)}\frac{c}{[c^d + c^b]^{\frac{1}{1+a}}}\right)^{1+a}, \tag{3.30}$$

then (3.29) has at least two nonnegative solutions.

To see that (3.29) has at least two nonnegative solutions, we will apply Theorem 3.1 with $\Phi(t) \equiv 1$, $f(t, x, |y|) = \mu(x^b + x^d)(1 + |y|^{-a})$, $h(x) = \mu(x^d + x^b)$, $g(|y|) = 1 + |y|^{-a}$, $r(|y|) = 0$, $g_1(x, y) = \mu(x^b)$ and $\Psi_{H,H'}(t) = \mu$. It is easy to see that

(3.1), (3.2), (3.4) and (3.5) hold. Since $I(z) = \int_0^z \frac{1}{1+u^{-a}}\,du \le \frac{1}{1+a}u^{1+a} = I_1(z)$, notice

$$\sup_{c\in(0,+\infty)} \frac{c}{I^{-1}\big(h(c)\int_0^1 \Phi(s)\,ds\big)} \ge \sup_{c\in(0,+\infty)} \frac{c}{I_1^{-1}(\mu(c^d+c^b))},$$

so (3.30) guarantees that (3.3) holds.

REMARK 3.1. Notice that (3.1)–(3.4) are only needed to guarantee the existence of $x_{0,1}$ in Theorem 3.1.

4. Multiple positive solutions to (1.1) with singularity at $x = 0$ and $x' = 0$. In this section our nonlinearity f may be singular at $x = 0$ and $x' = 0$. Throughout this section we will assume that the following conditions hold:

$$\Phi \in C[0,1] \text{ with } \Phi(t) > 0 \text{ on } (0,1) \tag{4.1}$$

$$\begin{cases} f : [0,1]\times(0,+\infty)\times(0,+\infty)\to(0,+\infty) \text{ is continuous with} \\ f(t,x,y) > 0 \text{ for } (t,x,y)\in[0,1]\times(0,+\infty)\times(0,+\infty), \\ f(t,x,y) \le [h(x)+w(x)][g(|y|)+r(|y|)] \text{ on } [0,1]\times(0,+\infty)\times(0,+\infty) \text{ with} \\ w>0, g>0 \text{ continuous and nonincreasing on } (0,+\infty), \\ \int_0^1 \Phi(s)r\big(k_0\frac{1}{s(1-s)}\big)\,ds < +\infty, \quad \int_0^{k_0} w(s)\,ds < +\infty, \quad \text{for all } k_0>0, \text{ and} \\ h\ge 0, \quad r\ge 0 \text{ continuous and nondecreasing on } [0,\infty) \end{cases} \tag{4.2}$$

$$\begin{cases} \sup_{c\in(0,+\infty)} \frac{c}{I^{-1}(ch(c)\|\Phi\|_1+\|\Phi\|_1\int_0^c w(s)\,ds)} > 1, \\ \text{where } I(z) = \int_0^z \frac{u}{g(u)+r(u)}\,du,\ z\in(0,+\infty), \end{cases} \tag{4.3}$$

$$\begin{cases} \text{for constants } H>0, H'>0 \text{ there exists a function } \psi_{H,H'} \\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ such that} \\ f(t,x,y) \ge \psi_{H,H'}(t) \text{ on } [0,1]\times(0,H]\times(0,H'] \end{cases} \tag{4.4}$$

and

$$\begin{cases} \text{there exists a } g_1 \in C((0,+\infty)\times(0,+\infty),(0,+\infty)) \text{ with} \\ f(t,x,y)\ge g_1(x,y), \forall (t,x,y)\in[0,1]\times(0,+\infty)\times(0,+\infty) \text{ such that} \\ \lim_{x\to+\infty}\frac{g_1(x,y)}{x} = +\infty \text{ uniformly for } y\in(0,+\infty). \end{cases} \tag{4.5}$$

THEOREM 4.1. *Suppose that (4.1)–(4.5) hold. Then (1.1) has at least two nonnegative solutions $x_{0,1}, x_{0,2} \in C^1[0,1]\cap C^2(0,1)$ with $x_{0,1}(t) > 0$ and $x_{0,2}(t) > 0$ on $(0,1)$.*

Proof. From (4.3) and the continuity of I^{-1} and h, choose an $R_1 > 0$, and a $\varepsilon > 0$ with $\varepsilon < \min\{\frac{R_1}{2}, 1\}$ and

$$\frac{R_1}{I^{-1}((R_1+\varepsilon)h(R_1+\varepsilon)\|\Phi\|_1 + \|\Phi\|_1\int_0^{R_1+\varepsilon} w(s)\,ds)} > 1. \tag{4.6}$$

Let $n_0 \in \{1,2,\ldots\}$ be chosen so that $\frac{2}{n_0} < \varepsilon$, and let $N_0 = \{n_0, n_0+1, \ldots\}$. For each $n \in N_0$, for $x \in P$, define

$$(A_n x)(t) = \int_0^1 G(t,s)\Phi(s)f\left(s, x(s)+\frac{1}{n}s+\frac{1}{n}, |x'(s)|+\frac{1}{n}\right)ds, \quad t\in[0,1]. \tag{4.7}$$

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Now (4.2) yields $f(t, x+\frac{1}{n}t+\frac{1}{n}, |x'|+\frac{1}{n}) \le [h(x+\frac{2}{n})+w(\frac{1}{n})][g(\frac{1}{n})+r(|x'|+\frac{1}{n})]$, which implies that (2.10) is true. The Lemma 2.6 guarantees that $A_n : P \to P$ is continuous and completely continuous with $A_n P \subseteq C^1[0,1]$ for each $n \in N_0$.

Let

$$\Omega_1 = \left\{x \in C_q^1[0,1] : \|x\| < R_1\right\}.$$

Choose $0 < a^* < b^* < 1$ and N^* as in (3.8). From (4.5), there exists an $R_2 > R_1$ such that

$$g_1(x, y) \ge N^* x, \quad \forall x \ge R_2, y \in (0, +\infty). \tag{4.8}$$

Let

$$\Omega_2 = \left\{x \in C_q^1[0,1] : \|x\| < \frac{4R_2}{a^*(1-b^*)}\right\}.$$

Then, for each $n \in N_0$, we claim that

$$\mu A_n x \ne x, \quad \forall \mu \in (0,1], \quad x \in P \cap \partial\Omega_1, \tag{4.9}$$

and

$$A_n x \not\le x, \quad \forall x \in P \cap \partial\Omega_2. \tag{4.10}$$

First we show that (4.9) is true. Suppose that there exists an $x_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $x_0 = \mu_0 A_n x_0$. Then

$$x_0''(t) + \mu_0 \Phi(t) f\left(t, x_0(t) + \frac{1}{n}t + \frac{1}{n}, |x_0'(t)| + \frac{1}{n}\right) = 0,\ t \in (0,1),\ x_0(0) = 0,\ x_0(1) = 0,$$

which means that $x_0(t) > 0$ on $(0,1)$ with $x_0(0) = x_0(1) = 0$ and $x_0'(t)$ is decreasing on $(0,1)$. Thus, there exists a unique $t_0 \in (0,1)$ with $x_0'(t_0) = 0$, $x_0'(t) > 0$ on $(0, t_0)$ and $x_0'(t) < 0$ on $(t_0, 1)$ and $\|x_0\|_1 = x_0(t_0)$. As a result (follow the argument used to prove (3.12), (3.13), (3.14), (3.15) and (3.16))

$$\begin{aligned} &\frac{-\left(x_0'(t)+\frac{1}{n}\right)x_0''(t)}{g\left(x_0'(t)+\frac{1}{n}\right)+r\left(x_0'(t)+\frac{1}{n}\right)} \\ &\quad \le \left[h\left(x_0(t)+\frac{1}{n}t+\frac{1}{n}\right)+w\left(x_0(t)+\frac{1}{n}t+\frac{1}{n}\right)\right]\left(x_0'(t)+\frac{1}{n}\right)\|\Phi\|_1, \quad \forall t \in (0, t_0), \end{aligned} \tag{4.11}$$

$$x_0'(t) \le I^{-1}\left(I(\varepsilon) + \|\Phi\|_1 h(R_1+\varepsilon)(R_1+\varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds\right), \tag{4.12}$$

$$\|x_0\|_1 = x_0(t_0) - x_0(0) \le I^{-1}\left(I(\varepsilon) + \|\Phi\|_1 h(R_1+\varepsilon)(R_1+\varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds\right), \tag{4.13}$$

$$\begin{aligned} &\frac{-\left(-x_0'(t)+\frac{1}{n}\right)x_0''(t)}{g\left(-x_0'(t)+\frac{1}{n}\right)+r\left(-x_0'(t)+\frac{1}{n}\right)} \\ &\quad \le \left[h(R_1+\varepsilon)+w\left(x_0(t)-\frac{1}{n}t+\frac{1}{n}\right)\right]\left(-x_0(t)+\frac{1}{n}t-\frac{1}{n}\right)'\|\Phi\|_1, \quad \forall t \in [t_0, 1), \end{aligned} \tag{4.14}$$

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$$-x_0'(t) < I^{-1}\left(I(\varepsilon) + \|\Phi\|_1 h(R_1+\varepsilon)(R_1+\varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds\right), \forall t \in [t_0, 1) \tag{4.15}$$

and

$$\sup_{t\in(0,1)} t(1-t)|x_0'(t)| \le I^{-1}(I(\varepsilon) + \|\Phi\|_1 h(R_1+\varepsilon)(R_1+\varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds), \forall t \in (0,1). \tag{4.16}$$

Now (4.16) and (4.13) guarantee that

$$R_1 = \max\{\|x_0\|_1, \|x_0\|_2\} \le I^{-1}\left(I(\varepsilon) + \|\Phi\|_1 h(R_1+\varepsilon)(R_1+\varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds\right),$$

which means that

$$\frac{R_1}{I^{-1}\big(I(\varepsilon) + \|\Phi\|_1 h(R_1+\varepsilon)(R_1+\varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds\big)} \le 1,$$

a contradiction to (4.6). Then, (4.9) is true.

Next we show that (4.10) is true. Suppose that there is an $x_0 \in P \cap \partial\Omega_2$ with $x_0 \ge A_n x_0$. Then by the definition of the cone and Lemma 2.4, one has $x_0(t) \ge t(1-t)\|x_0\|_1 \ge a^*(1-b^*)\frac{1}{4}\|x_0\| = a^*(1-b^*)\frac{1}{4}\frac{4R_2}{a^*(1-b^*)} = R_2, \quad \forall t \in [a^*, b^*]$. Clearly for $t \in [a^*, b^*]$, $x_0(t) + \frac{1}{n}t + \frac{1}{n} \ge R_2, \quad \forall t \in [a^*, b^*]$. Then, for $t \in [a^*, b^*]$, from (4.8), one has

$$\begin{aligned}
x_0(t) \ge (A_n x_0)(t) &\ge \int_{a^*}^{b^*} G(t,s)\Phi(s) g_1\left(x_0(s) + \frac{1}{n}s + \frac{1}{n}, |x_0'(s)| + \frac{1}{n}\right) ds \\
&\ge \int_{a^*}^{b^*} G(t,s)\Phi(s) N^*\left(x_0(s) + \frac{1}{n}s + \frac{1}{n}\right) ds \ge \int_{a^*}^{b^*} G(t,s)\Phi(s)\,ds N^* R_2 \\
&= a^*(1-b^*)\frac{1}{4}\min_{t\in[a^*,b^*]} \int_{a^*}^{b^*} G(t,s)\Phi(s)\,ds N^* \frac{4R_2}{a^*(1-b^*)} > \frac{4R_2}{a^*(1-b^*)},
\end{aligned}$$

which implies that $\|x_0\| \ge \|x_0\|_1 > \frac{4R_2}{a^*(1-b^*)}$, a contradiction to $x_0 \in P \cap \partial\Omega_2$. Then, (4.10) is true.

Lemma 2.2 and Lemma 2.3 guarantee that

$$i(A_n, P \cap \Omega_1, P) = 1, \quad i(A_n, P \cap \Omega_2, P) = 0. \tag{4.17}$$

Thus

$$i(A_n, P \cap (\Omega_2 - \overline{\Omega}_1), P) = -1, \quad n \in N_0. \tag{4.18}$$

As a result, for each $n \in N_0$, there exist $x_{n,1} \in P \cap \Omega_1$ and $x_{n,2} \in P \cap (\Omega_2 - \overline{\Omega}_1)$ such that $x_{n,1} = A_n x_{n,1}$ and $x_{n,2} = A_n x_{n,2}$.

Now we consider $\{x_{n,1}\}_{n\in N_0}$ and $\{x_{n,2}\}_{n\in N_0}$. It is easy to see that the functions belonging to $\{x_{n,1}(t)\}$ are uniformly bounded on $[0, 1]$ with $\max_{t\in[0,1]} |x_{n,1}(t)| \le R_1$, $n \in N_0$.

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For each $n \in N_0$ one has

$$x''_{n,1}(t)+\Phi(t)f\left(t,\, x_{n,1}(t)+\frac{1}{n}t+\frac{1}{n},\, |x'_{n,1}(t)|+\frac{1}{n}\right)=0,\ t\in(0,1),\ x_{n,1}(0)=0,\ x_{n,1}(1)=0, \tag{4.19}$$

which means that $x_{n,1}(t) > 0$ on $(0, 1)$ with $x_{n,1}(0) = x_{n,1}(1) = 0$ and $x'_{n,1}(t)$ is decreasing on $(0, 1)$ and there exists a unique $t_n \in (0, 1)$ with $x'_{n,1}(t_n) = 0$, $x'_{n,1}(t_n) > 0$ on $(0, t_n)$ and $x'_{n,1}(t) < 0$ on $(t_n, 1)$ and $\|x_{n,1}\|_1 = x_{n,1}(t_{n,1})$. Following the ideas used to prove (4.11) and (4.14) yields

$$\begin{aligned}&\frac{-\big(|x'_{n,1}(t)|+\frac{1}{n}\big)x''_{n,1}(t)}{g\big(|x'_{n,1}(t)|+\frac{1}{n}\big)+r\big(|x'_{n,1}(t)|+\frac{1}{n}\big)}\\&\quad\le\left[h\left(x_{n,1}(t)+\frac{1}{n}t+\frac{1}{n}\right)+w\left(x_{n,1}(t)+\frac{1}{n}t+\frac{1}{n}\right)\right]\left(|x'_{n,1}(t)|+\frac{1}{n}\right)\|\Phi\|_1,\\&\qquad\qquad\qquad\qquad\qquad\qquad\forall t\in(0,t_n)\cup(t_n,1).\end{aligned}\tag{4.20}$$

A similar argument to that used to prove (4.12) and (4.15) yields

$$\sup_{n\ge n_0}\sup_{t\in(0,1)}|x'_{n,1}(t)|\le I^{-1}\left(I(\varepsilon)+\|\Phi\|_1h(R_1+\varepsilon)(R_1+\varepsilon)+\|\Phi\|_1\int_0^{R_1+\varepsilon}w(s)\,ds\right),$$

i.e., the functions belonging to $\{x'_{n,1}(t)\}$ are uniformly bounded on $[0, 1]$, which guarantees that the functions belonging to $\{x_{n,1}(t)\}$ are equicontinuous on $[0, 1]$.

Next we show that the functions belonging to $\{x'_{n,1}(t)\}$ are equicontinuous on $[0, 1]$. For any $t_1, t_2 \in [0, 1]$, (4.20) yields

$$\begin{aligned}&\left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)\right|\\&\quad\le\|\Phi\|_1\left[h(R_1+\varepsilon)(|x_{n,1}(t_1)-x_{n,1}(t_2)|+|t_1-t_2|)+\left|\int_{x_{n,1}(t_1)-\frac{1}{n}t_1+\frac{1}{n}}^{x_{n,1}(t_2)-\frac{1}{n}t_2+\frac{1}{n}}w(s)\,ds\right|\right],\end{aligned}$$

if $t_2 > t_1 > t_n$ and

$$\begin{aligned}&\left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)\right|\\&\quad\le\|\Phi\|_1|\left[h(R_1+\varepsilon)(|x_{n,1}(t_1)-x_{n,1}(t_2)|+|t_1-t_2|)+\left|\int_{x_{n,1}(t_1)+\frac{1}{n}t_1+\frac{1}{n}}^{x_{n,1}(t_2)+\frac{1}{n}t_2+\frac{1}{n}}w(s)\,ds\right|\right],\end{aligned}$$

if $t_2 < t_1 < t_n$, which guarantee that

$$\begin{aligned}&\left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)\right|\\&\quad\le\left|I\left(|x'_{n,1}(t_2)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_n)|+\frac{1}{n}\right)\right|+\left|I\left(|x'_{n,1}(t_1)|+\frac{1}{n}\right)-I\left(|x'_{n,1}(t_n)|+\frac{1}{n}\right)\right|\\&\quad\le\|\Phi\|_1|\left[h(R_1+\varepsilon)(|x'_{n,1}(t_2)-x'_{n,1}(t_n)|+|t_2-t_n|)+\left|\int_{x_{n,1}(t_n)-\frac{1}{n}t_n+\frac{1}{n}}^{x_{n,1}(t_2)-\frac{1}{n}t_2+\frac{1}{n}}w(s)\,ds\right|\right]\\&\qquad+\|\Phi\|_1|\left[h(R_1+\varepsilon)(|x'_{n,1}(t_1)-x'_{n,1}(t_n)|+|t_1-t_n|)+\left|\int_{x_{n,1}(t_n)+\frac{1}{n}t_n+\frac{1}{n}}^{x_{n,1}(t_1)+\frac{1}{n}t_1+\frac{1}{n}}w(s)\,ds\right|\right],\end{aligned}$$

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if $t_2 > t_n > t_1$ or

$$\begin{aligned}
&\left| I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| \\
&\quad \le \|\Phi\|_1 \left[h(R_1+\varepsilon)(|x'_{n,1}(t_2) - x'_{n,1}(t_n)| + |t_2 - t_n|) + \left| \int_{x_{n,1}(t_n)+\frac{1}{n}t_n+\frac{1}{n}}^{x_{n,1}(t_2)+\frac{1}{n}t_2+\frac{1}{n}} w(s)\,ds \right| \right] \\
&\qquad + \|\Phi\|_1 \left[h(R_1+\varepsilon)(|x'_{n,1}(t_1) - x'_{n,1}(t_n)| + |t_1 - t_n|) + \left| \int_{x_{n,1}(t_n)-\frac{1}{n}t_n+\frac{1}{n}}^{x_{n,1}(t_1)-\frac{1}{n}t_1+\frac{1}{n}} w(s)\,ds \right| \right],
\end{aligned}$$

if $t_2 < t_n < t_1$.

The uniform continuity of $\int_0^z w(s)\,ds$ on bounded interval implies that for any $\varepsilon' > 0$, there is a $\delta > 0$ such that $|\int_{z_1}^{z_2} w(s)\,ds| < \frac{\varepsilon'}{4\|\Phi\|_1}$, $|z_1 - z_2| < \delta$, $z_1, z_2 \in [0, R_1+\varepsilon]$. Also the equicontinuity of $\{x_{n,1}(t)\}$ implies that there is a $\min\{\frac{\delta}{2}, \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\} > \delta' > 0$ such that $|x_{n,1}(t_1) - x_{n,1}(t_2)| < \min\{\frac{\delta}{2}, \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\}$, $\forall n \in N_0$, $|t_1 - t_2| < \delta'$, $t_1, t_2 \in [0, 1]$. Consequently,

$$\begin{aligned}
&\left| I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| \\
&\quad \le \|\Phi\|_1 \left[h(R_1+\varepsilon)\left(\frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)} + \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\right) + \frac{\varepsilon'}{4\|\Phi\|_1}\right] < \frac{\varepsilon'}{2},
\end{aligned}$$

if $t_2 > t_1 > t_n$ and

$$\begin{aligned}
&\left| I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| \\
&\quad \le \|\Phi\|_1 \left[h(R_1+\varepsilon)\left(\frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)} + \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\right) + \frac{\varepsilon'}{4\|\Phi\|_1}\right] < \frac{\varepsilon'}{2},
\end{aligned}$$

if $t_2 < t_1 < t_n$, which guarantee that

$$\begin{aligned}
&\left| I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| \\
&\quad \le \|\Phi\|_1 \left[h(R_1+\varepsilon)\left(\frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)} + \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\right) + \frac{\varepsilon'}{4\|\Phi\|_1}\right] \\
&\qquad + \|\Phi\|_1 \left[h(R_1+\varepsilon)\left(\frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)} + \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\right) + \frac{\varepsilon'}{4\|\Phi\|_1}\right] < \varepsilon',
\end{aligned}$$

if $t_2 > t_n > t_1$ or

$$\begin{aligned}
&\left| I\left(|x'_{n,1}(t_2)| + \frac{1}{n}\right) - I\left(|x'_{n,1}(t_1)| + \frac{1}{n}\right)\right| \\
&\quad \le \|\Phi\|_1 \left[h(R_1+\varepsilon)\left(\frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)} + \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\right) + \frac{\varepsilon'}{4\|\Phi\|_1}\right] \\
&\qquad + \|\Phi\|_1 [h(R_1+\varepsilon)\left(\frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)} + \frac{\varepsilon'}{8\|\Phi\|_1 h(R_1+\varepsilon)}\right) + \frac{\varepsilon'}{4\|\Phi\|_1}\Bigg] < \varepsilon',
\end{aligned}$$

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if $t_2 < t_n < t_1$, which means that the functions belonging to $\{I(|x'_{n,1}(t)| + \frac{1}{n})\}$ are equicontinuous on $[0, 1]$. By the continuity of $I^{-1}(z)$, we know that the functions belonging to $\{|x'_{n,1}(t)| + \frac{1}{n}\}$ are equicontinuous on $[0, 1]$, which implies the functions belonging to $\{|x'_{n,1}(t)|\}$ are equicontinuous on $[0, 1]$. Then, for any $\varepsilon' > 0$, there is a $\delta > 0$ such that $\||x'_{n,1}(t_1)| - |x'_{n,1}(t_2)|\| < \varepsilon'$, $\forall |t_1 - t_2| < \delta$, $t_1, t_2 \in [0, 1]$, which implies $|x'_{n,1}(t_1) - x'_{n,1}(t_2)| = \||x'_{n,1}(t_1)| - |x'_{n,1}(t_2)|\| < \varepsilon'$, $\forall |t_1 - t_2| < \delta$, $t_1, t_2 \le t_n \in [0, 1]$, $|x'_{n,1}(t_1) - x'_{n,1}(t_2)| = \||x'_{n,1}(t_1)| - |x'_{n,1}(t_2)|\| < \varepsilon'$, $\forall |t_1 - t_2| < \delta$, $t_1, t_2 \ge t_n \in [0, 1]$ and

$$\begin{aligned}
|x'_{n,1}(t_1) - x'_{n,1}(t_2)| &\le |x'_{n,1}(t_1) - x'_{n,1}(t_n)| + |x'_{n,1}(t_n) - x'_{n,1}(t_2)| \\
&= \||x'_{n,1}(t_1)| - |x'_{n,1}(t_n)|\| + \||x'_{n,1}(t_n)| - |x'_{n,1}(t_2)|\| \\
&< 2\varepsilon', \quad \forall |t_1 - t_2| < \delta,\ t_1 < t_n < t_2 \in [0, 1], \quad \text{or} \quad t_2 < t_n < t_1 \in [0, 1],
\end{aligned}$$

which means that the functions belonging to $\{x'_{n,1}(t)\}$ are equicontinuous on $[0, 1]$.

The Arzela-Ascoli Theorem guarantees that $\{x_{n,1}\}$ is relatively compact in $C^1[0, 1]$, i.e., that there is a convergence subsequence $\{x_{n_j,1}\}$ of $\{x_{n,1}\}$ such that $\lim_{j\to+\infty} x_{n_j,1} = x_{0,1} \in C^1[0, 1]$ with $x'_{n_j,1}(t_{n_j}) = 0$. Without loss of generality, we assume that $t_{n_j} \to t_0 \in [0, 1]$ as $j \to +\infty$. Clearly $x'_{0,1}(t_0) = 0$ (keep also in mind that $\{x'_{n_j,1}\}$ converges to $x'_{0,1}$ uniformly on $[0, 1]$ and $x'_{0,1}$ is continuous on $[0, 1]$).

Now we show that

$$t_0 \in (0, 1), \quad \text{and} \quad x'_{0,1}(t) \neq 0 \quad \text{for all } t \in (0, 1)\backslash\{t_0\}. \tag{4.21}$$

Suppose $t_0 = 0$. Let $R' = I^{-1}(I(\varepsilon) + \|\Phi\|_1 h(R_1 + \varepsilon)(R_1 + \varepsilon) + \|\Phi\|_1 \int_0^{R_1+\varepsilon} w(s)\,ds) + 1$. From (4.4), there is a $\Psi_{R_1+\varepsilon,R'} \in C[0, 1]$ with $\Psi_{R_1+\varepsilon,R'}(t) > 0$ for all $t \in (0, 1)$ such that

$$\begin{aligned}
-x''_{n_j,1}(t) &= \Phi(t) f\left(t, x_{n_j,1}(t) + \frac{1}{n_j}t + \frac{1}{n_j}, |x'_{n_j,1}(t)| + \frac{1}{n_j}\right) \\
&\ge \Phi(t)\Psi_{R_1+\varepsilon,R'}(t), \qquad t \in (0, 1).
\end{aligned} \tag{4.22}$$

Then $x'_{n_j,1}(t_{n_j}) - x'_{n_j,1}(t) \ge \int_{t_{n_j}}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds$, $t \in (t_{n_j}, 1)$. Letting $j \to +\infty$, we have

$$-x'_{0,1}(t) \ge \int_{t_0}^t \Phi(s)\Phi_{R_1+\varepsilon,R'}(s)\,ds = \int_0^t \Phi(s)\Phi_{R_1+\varepsilon,R'}(s)\,ds, \qquad t \in (0, 1),$$

which means $x_{0,1}(t)$ is decreasing on $(0, 1)$, a contradiction to $x_{0,1}(0) = x_{0,1}(1)$. Similarly, we get $t_0 < 1$.

Moreover, (4.22) implies that

$$\begin{aligned}
x'_{n_j,1}(t_{n_j}) - x'_{n_j,1}(t) &\ge \int_{t_{n_j}}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds,\ t \in (t_{n_j}, 1), \\
x'_{n_j,1}(t) - x'_{n_j,1}(t_{n_j}) &\ge \int_t^{t_{n_j}} \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds,\ t \in (0, t_{n_j}).
\end{aligned}$$

Letting $j \to +\infty$, one gets

$$\begin{aligned}
-x'_{0,1}(t) \ge \int_{t_0}^t \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds,\ \ t \in (t_0, 1), \quad \text{and } x'_{0,1}(t) \\
\ge \int_t^{t_0} \Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds, \qquad t \in (0, t_0),
\end{aligned}$$

which implies that $x'_{0,1}(t) \neq 0$ for all $t \neq t_0$. Hence, (4.21) is true.

Consequently, for any $\frac{\min\{t_0,1-t_0\}}{2} > \varepsilon' > 0$, there exists an $N > 0$ such that $t_{n_j} \in [t_0 - \frac{\varepsilon'}{2}, t_0 + \frac{\varepsilon'}{2}]$ for all $j \geq N$, which guarantees that

$$\min\left\{\inf_{j\geq N}\inf_{t\in(0,t_0-\varepsilon']}|x'_{n_j}(t)|,\ \inf_{j\geq N}\inf_{t\in[t_0+\varepsilon',1)}|x'_{n_j}(t)|\right\} = \delta_0 > 0.$$

On the other hand, Lemma 2.5 yields

$$\begin{aligned} x_{n_j,1}(t) &\geq \int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds \\ &\geq t(1-t)\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds \\ &\geq \varepsilon'^2\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds, \quad t\in[\varepsilon',1-\varepsilon']. \end{aligned}$$

Let $\delta'_0 = \varepsilon'^2\max_{t\in[0,1]}\int_0^1 G(t,s)\Phi(s)\Psi_{R_1+\varepsilon,R'}(s)\,ds$. Then,

$$\begin{aligned} &\Phi(t)\left|f\left(t, x_{n_j,1}(t) + \frac{1}{n_j}t + \frac{1}{n_j},\ |x'_{n_j,1}(t)| + \frac{1}{n_j}\right)\right| \\ &\quad\leq [h(R_1+1) + w(\delta'_0)][g(\delta_0) + r(R')]\Phi(t), \quad t\in[\varepsilon', t_0-\varepsilon']\cup[t_0+\varepsilon', 1-\varepsilon']. \end{aligned}$$

Hence, the Lebesgue Dominated Convergence Theorem guarantees that

$$\begin{aligned} x'_{0,1}(t) - x'_{0,1}(t_0+\varepsilon') &= \lim_{j\to+\infty}\left[x'_{n_j,1}(t) - x'_{n_j,1}(t_0+\varepsilon')\right] \\ &= \lim_{j\to+\infty}\int_{t_0+\varepsilon'}^t \Phi(s)f\left(s, x_{n_j,1}(s) + \frac{1}{n_j}s + \frac{1}{n_j},\ |x'_{n_j,1}(s)| + \frac{1}{n_j}\right)ds \\ &= \int_{t_0+\varepsilon'}^t \Phi(s)f(s, x_{0,1}(s), |x'_{0,1}(s)|)\,ds, \quad t\in[t_0+\varepsilon', 1-\varepsilon'] \end{aligned}$$

and similarly

$$x'_{0,1}(t) - x'_{0,1}(t_0-\varepsilon') = \int_{t_0-\varepsilon'}^t \Phi(s)f(s, x_{0,1}(s), |x'_{0,1}(s)|)\,ds, \quad t\in[\varepsilon', t_0-\varepsilon'].$$

Consequently, $-x''_{0,1}(t) = \Phi(t)f(t, x_{0,1}(t), |x'_{0,1}(t)|)$, $t\in(\varepsilon', t_0-\varepsilon')\cup(t_0+\varepsilon', 1-\varepsilon')$. Since ε' is arbitrary, we have $-x''_{0,1}(t) = \Phi(t)f(t, x_{0,1}(t), |x'_{0,1}(t)|)$, $t\in(0,t_0)\cup(t_0,1)$. In addition $x_{0,1}(0) = x_{0,1}(1) = 0$, and $x_{0,1}$ is a solution of (1.1) with $x_{0,1}(t) > 0$ on $(0,1)$.

For the set $\{x_{n,2}\}_{n\in N_0} \subseteq (\Omega_2 - \overline{\Omega}_1)\cap P$, a similar proof yields a convergent subsequence $\{x_{n_i,2}\}$ of $\{x_{n,2}\}$ with $\lim_{i\to+\infty} x_{n_i,2} = x_{0,2} \in C^1[0,1]$. Also $x_{0,2}$ is a nonnegative solution of (1.1) with $x_{0,2}(t) > 0$ on $(0,1)$ and $R_1 < \|x_{0,2}\| < \frac{4R_2}{a^*(1-b^*)}$.

Consequently, (1.1) has at least two different nonnegative solutions $x_{0,1}$ and $x_{0,2}$ with $\|x_{0,1}\| < R_1 < \|x_{0,2}\|$. □

EXAMPLE 4.1. Consider the boundary value problems

$$\begin{cases} x'' + \mu[1 + |x'|^e + |x'|^{-a}][y^b + y^{-d} + 1] = 0, & t\in(0,1) \\ x(0) = 0, \quad x(1) = 0, \end{cases} \tag{4.23}$$

with $0 \le a$, $0 \le e < 1$, $b > 1$, $0 < d < 1$ and $\mu > 0$. If

$$\mu < \frac{1}{2+a}\left(\sup_{c\in(0,+\infty)} \frac{c}{[c + c^{b+1} + c^{1-d}/(1-d)]^{\frac{1}{2+a}}}\right)^{2+a}, \tag{4.24}$$

then (4.23) has at least two nonnegative solutions.

REMARK 4.1. Notice that (4.1)–(4.4) are only needed to guarantee the existence of $x_{0,1}$ in Theorem 4.1.

REFERENCES

1. R. P. Agarwal, D. O'Regan and V. Lakshmikantham, Nonuniform nonresonance at the first eigenvalue for singular boundary value problems with sign changing nonlinearities, *J. Math. Anal. Appl.* **274** (2002), 404–423.

2. R. P. Agarwal and D. O'Regan, Existence theory for single and multiple solutions to singular positone boundary value problems, *J. Differential Equations* **175** (2001), 393–414.

3. A. Capielto and W. Dambrocio, Multiplicity results for some two-point superlinear asymmetric boundary value problems, *Nonlinear Analysis* **38** (1999), 869–896.

4. K. Coudreau, A multiplicity result for a nonlinear boundary value problem, *J. Math. Anal. Appl.* **218** (1998), 395–408.

5. D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones* (Academic Press, Inc., 1988).

6. J. Henderson and H. B. Thompson, Existence of multiple solutions for second order boundary value problems, *J. Differential Equations* **166** (2000), 443–454.

7. P. Kelevedjiev, Existence of positive solutions to a singular second order boundary value problems, *Nonlinear Analysis* **50** (2002), 1107–1118.

8. M. Meehan and D. O'Regan, Multiple nonnegative solutions of nonlinear integral equations on compact and semi-infinite intervals, *Applicable Analysis* **74** (2000), 413–427.

9. D. O'Regan, *Theory of singular boundary value problems* (World Scientific, Singapore, 1994).

10. D. O'Regan, *Existence theory for nonlinear ordinary differential equations* (Kluwer Academic Publishers, 1997).

11. S. Staněk, Positive solutions of singular positone Dirichlet boundary value problems, *Mathematical and Computer Modelling* **33** (2001), 341–351.

12. I. Yermachenko and F. Sadyrbaev, Types of solutions and multiplicity results for two-point nonlinear boundary value problems, *Nonlinear Analysis*, in press.