J. Inst. Math. Jussieu (2025), **24**(1), 17–39
doi:10.1017/S1474748024000197

TWISTED GAN–GROSS–PRASAD CONJECTURE FOR CERTAIN TEMPERED L-PACKETS

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(Received 25 August 2022; revised 9 April 2024; accepted 9 April 2024; first published online 24 May 2024)

Abstract In this paper, we investigate the twisted GGP conjecture for certain tempered representations using the theta correspondence and establish some special cases, namely when the L-parameter of the unitary group is the sum of conjugate-dual characters of the appropriate sign.

1. Problem, conjecture and results

In a recent paper [GGP23], a twisted version of the Gan–Gross–Prasad (GGP) conjecture was formulated in the context of skew-Hermitian spaces and their associated unitary groups over local and global fields. Some evidence was provided in [GGP23] for the local twisted conjecture, such as in low rank situations and for unitary principal series representations. The purpose of this paper is to provide further affirmative evidence by establishing the local conjecture for a family of tempered L-packets of unitary groups using the technique of theta correspondence. Let us recall the setup and conjecture of [GGP23] in greater precision and formulate our main result.

1.1. Biquadratic extension

Let F be a non-Archimedean local field of characteristic 0 and $E \neq K$ two distinct quadratic field extensions of F. Let $L = E \otimes_F K$ so that L is a biquadratic extension of F. We thus have the picture:

2020 Mathematics subject classification: Primary 11F27, 11F70
Secondary 22E50

$$\begin{tikzcd}[column sep=small] & L \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\tau"] & \\ K \arrow[dr, dash, "\tau"'] & & E \arrow[dl, dash, "\sigma"] \\ & F & \end{tikzcd}$$

In particular, we have set:

$$\mathrm{Gal}(E/F) \simeq \mathrm{Gal}(L/K) \simeq \langle\sigma\rangle, \quad \textit{and} \quad \mathrm{Gal}(K/F) \simeq \mathrm{Gal}(L/E) \simeq \langle\tau\rangle.$$

We also fix a nontrivial additive character ψ_F of F and set $\psi_K = \psi_F \circ \mathrm{Tr}_{K/F}$. In this paper, when we talk about Weil representations or theta correspondence, we always use the additive character ψ_F or ψ_K (see Section 2.1).

1.2. Skew-Hermitian spaces

Let V be an n-dimensional skew-Hermitian space over E. There are exactly two such spaces, which are distinguished by their sign

$$\epsilon(V) = \omega_{E/F}(\delta^{-n} \cdot \mathrm{disc} V),$$

where $\mathrm{disc} V = (-1)^{n(n-1)/2} \cdot \det V$, and δ is a fixed trace zero element in $E^\times$. As observed in [GGP23, Lem. 8.1], the scalar extension $V_K = V \otimes_F K$ is a distinguished split skew-Hermitian space over L whose isomorphism class is independent of the choice of V. In particular, if we continue to use the trace zero element $\delta \in L^\times$ to define the sign of V_K, then we always have $\epsilon(V_K) = +1$.

1.3. Twisted GGP problem

We come now to the restriction problem to be studied. For the skew-Hermitian space V over E, we have the Weil representation $\omega_{V,\mu}$, where μ is a conjugate-symplectic character of $E^\times$. Then we are interested in determining

$$m_V(\pi,\mu) = \dim \mathrm{Hom}_{\mathrm{U}(V)}(\pi, \omega_{V,\mu}) \quad \textit{for } \pi \in \mathrm{Irr}(\mathrm{U}(V_K)).$$

Here is the main local conjecture for the twisted GGP problem:

Conjecture 1.1.

(1) *For each $\pi \in \mathrm{Irr}(\mathrm{U}(V_K))$, $m_V(\pi,\mu) \leq 1$.*

(2) *Let M be a generic L-parameter of $\mathrm{U}(V_K)$ with associated L-packet Π_M. Then*

$$\sum_V \sum_{\pi \in \Pi_M} m_V(\pi,\mu) = 1,$$

where the first sum runs over the two skew-Hermitian spaces over E of dimension n, and the second runs over the L-packet Π_M.

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(3) *The unique V_0 which has nonzero contribution to the sum in* (2) *is characterized by*

$$\epsilon(V_0) = \epsilon\left(\frac{1}{2}, \mathrm{As}^{+}_{L/E}(M) \otimes \mu^{-1}, \psi_{E,\delta}\right) \cdot \omega_{K/F}\left(\delta^2\right)^{n(n-1)/2},$$

where δ is the fixed trace zero element in $E^\times$ (used in the definition of $\epsilon(V_0)$), and $\psi_{E,\delta} = \psi_F(\mathrm{Tr}_{E/F}(\delta\cdot\ ))$.

(4) *The unique $\pi \in \Pi_M$ which has nonzero contribution to the sum in* (2) *corresponds via local Langlands correspondence (LLC) (with respect to the Whittaker datum of $\mathrm{U}(V_K)$ associated to ψ_K) to the character of local component group $A_M = \prod_{i\in I} \mathbb{Z}/2\mathbb{Z}\cdot a_i$ given by*

$$\begin{aligned}\eta(a_i) &= \epsilon\left(\frac{1}{2}, \mathrm{Ind}_L^E\left({}^{\tau}M_i \otimes (M/M_i)\right)\cdot \mu^{-1}, \psi_{E,\delta}\right)\\ &= \epsilon\left(\frac{1}{2}, [\mathrm{As}(M_i) + \mathrm{As}(M) + \mathrm{As}(M/M_i)]\cdot \mu^{-1}, \psi_{E,\delta}\right),\end{aligned}$$

where M_i is the irreducible constituent of M corresponding to $a_i \in A_M$.

We remark that [GGP23] also formulated a conjecture in the case $E = K$ and showed that, in this case, the conjecture can be reduced to the case of discrete series representations of $\mathrm{U}(V_K) \simeq \mathrm{GL}(V)$. However, we do not deal with the case $E = K$ in this paper.

In [GGP23, Sect. 9 & 10], the three authors have proved that:

Theorem 1.2.

(1) *Conjecture 1.1 holds if $n \leq 2$.*

(2) *Conjecture 1.1(1)–(3) hold for unitary principal series representations (induced from the Borel subgroup), and (4) holds as well if the unitary principal series is irreducible.*

1.4. Main result

Our main result is the following theorem.

Theorem 1.3. *Let M be a tempered L-parameter for $\mathrm{U}(V_K)$ of the form*

$$M = M_1 + \cdots + M_n$$

with each M_i one-dimensional and conjugate self-dual of parity $(-1)^{n-1}$. Then Conjecture 1.1 holds for M.

The parity condition on each M_i is equivalent to requiring that the L-parameter M is of good parity. Note that, though these tempered L-parameters M are maximally reducible and hence not the most general in the p-adic case, they are the ones whose L-packets are of maximal size. Hence, in some sense, they provide the most stringent test for Conjecture 1.1.

Recall that by properties of the LLC, tempered L-packets can be constructed using irreducible parabolic induction from good parity L-packets. An immediate corollary of our result (combining with [GGP23, Thm. 10.1]) is that we may complete Theorem 1.2(2) above:

Corollary 1.4. *Conjecture 1.1 holds for the tempered L-packets consisting of the constituents of unitary principal series representations.*

1.5. Idea of proof

The main tool for the proof of Theorem 1.3 is the theta correspondence. Using theta correspondence, we shall effectively show that Theorem 1.3 for the case $\dim V = n+1$ can be reduced to the case for $\dim V = n$. In this way, for the type of tempered L-parameters M considered in Theorem 1.3, we may use theta correspondence to successively strip off the irreducible summands M_i one at a time and reduce the conjecture for such M's to the case when $\dim V = 1$. In fact, since the conjecture has been shown for $\dim V \leq 2$, we could have formulated a slightly more general main result. We content ourselves with just the following corollary:

Corollary 1.5. *Conjecture 1.1 holds for all endoscopic tempered L-packets of* $\mathrm{U}(V_K)$ *when* $\dim V = 3$*.*

This is because all endoscopic tempered L-packets of U_3 can be constructed by theta lifting from tempered L-packets of U_2.

The rest of the paper is devoted to the proof of Theorem 1.3. In §2, we study a local theta lift of a Weil representation of a unitary group to the edge of the stable range. The main point here is to show that the resulting big theta lift is irreducible. Then in §3, we show how the conjecture in dimension $n+1$ can be reduced to that in dimension n by invoking two seesaw arguments. In the proofs, we have made use of the so-called Adams' conjecture, which describes the theta correspondence in terms of (conjectural) A-packets. But our result is not conditional on the construction of A-packets; we refer the readers to Remark 2.2 for details.

2. Weil representations

In this section, we examine the Weil representation $\omega_{V,\mu}$ and investigate its behavior under the theta correspondence.

2.1. Local theta correspondence

We first recall the basic setup of the local theta correspondence. Let $F \subset E$ be a quadratic extension of non-Archimedean local fields, V an skew-Hermitian space of dimension n and W an Hermitian space of dimension m. We shall use the symbol $\mathcal{H}$ (resp. $\mathcal{H}'$) to denote the skew-Hermitian (resp. Hermitian) hyperbolic plane.

To consider the theta correspondence for the reductive dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$, one requires some additional data:

- a nontrivial additive character ψ_F of F;
- a pair of characters χ_V and χ_W of $E^\times$ such that

$$\chi_V\big|_{F^\times} = \omega_{E/F}^{\dim V} \quad \textit{and} \quad \chi_W\big|_{F^\times} = \omega_{E/F}^{\dim W}.$$

To elaborate, the tensor product $V \otimes W$ has a natural symplectic form, which induces a natural map

$$\mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes W).$$

One has the metaplectic S^1-cover $\mathrm{Mp}(V \otimes W)$ of $\mathrm{Sp}(V \otimes W)$, and the character ψ_F determines a Weil representation ω_{ψ_F} of $\mathrm{Mp}(V \otimes W)$. The datum (ψ_F, χ_V, χ_W) then allows one to specify a splitting of the metaplectic cover over $\mathrm{U}(V) \times \mathrm{U}(W)$ following [Kud94]. Hence, we have a Weil representation $\omega = \omega_{V,W}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$.

As explicated in [Kud94] and [HKS96], the splitting over $\mathrm{U}(V)$ is determined by (ψ_F, χ_W), whereas that of $\mathrm{U}(W)$ by (ψ_F, χ_V). In particular, taking W such that $\dim W = 1$ and $\chi_W = \mu$ a conjugate symplectic character of $E^\times$, one gets a splitting over $\mathrm{U}(V)$ associated to (ψ_F, μ), and thus a Weil representation $\omega_{V,\mu}$ of $\mathrm{U}(V)$, which is the one appearing in the main conjecture.

Given an irreducible representation π of $\mathrm{U}(V)$, the maximal π-isotypic quotient of ω is of the form

$$\Theta(\pi) \boxtimes \pi$$

for some smooth representation $\Theta(\pi)$ of $\mathrm{U}(W)$ of finite length. By the Howe duality [Wal90] [GT16a] [GT16b], we have:

- The maximal semisimple quotient $\theta(\pi)$ of $\Theta(\pi)$ is irreducible if $\Theta(\pi)$ is nonzero;
- If $\pi_1 \not\simeq \pi_2$ are two nonisomorphic irreducible smooth representations of $\mathrm{U}(V)$ such that both $\theta(\pi_1)$ and $\theta(\pi_2)$ are nonzero, then $\theta(\pi_1) \not\simeq \theta(\pi_2)$.

2.2. A refinement of Adams' conjecture

Next, we give a description of the theta correspondence in terms of A-parameters. We fix a nontrivial additive character ψ_F once and for all. Assume that

$$m = \dim W \geq n = \dim V \geq 1.$$

Fix a pair of splitting characters (χ_V, χ_W), and consider the theta correspondence between $\mathrm{U}(V) \times \mathrm{U}(W)$ with respect to it.

Let Ψ be a local A-parameter of $\mathrm{U}(V)$. If we write it as a sum of irreducible subrepresentations

$$\Psi = \sum_i \rho_i S_{a_i} \boxtimes S_{b_i},$$

we say that Ψ is of good parity if $\rho_i S_{a_i} \boxtimes S_{b_i}$ is conjugate self-dual of parity $(-1)^{n-1}$ for all i. Here, we are following Atobe–Gan's notation [AG17] on irreducible representations of the Weil–Deligne group $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$; we omit the tensor symbol between ρ_i and S_{a_i} to distinguish finite-dimensional representations of the Weil–Deligne $\mathrm{SL}_2(\mathbb{C})$ and the Arthur $\mathrm{SL}_2(\mathbb{C})$.

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Theorem 2.1.

(1) *Assume that* Ψ *is of good parity and*

$$m-n \geq \max_i \left\{ b_i - a_i + 1 \mid \rho_i \simeq \chi_W \right\}.$$

Let π *be an irreducible unitary representation in the local A-packet* $\Pi_\Psi(\mathrm{U}(V))$*. Then the theta lift* $\theta(\pi)$ *of* π *to* $\mathrm{U}(W)$ *lies in the local A-packet* $\Pi_{\theta(\Psi)}(\mathrm{U}(W))$ *if it is nonzero, where*

$$\theta(\Psi) = \Psi\chi_W^{-1}\chi_V + \chi_V \boxtimes S_{m-n}.$$

(2) *Moreover, if we further assume that*

$$m-n > \max_i \left\{ b_i + a_i - 1 \mid \rho_i \simeq \chi_W \right\},$$

then $\theta(\pi)$ *must be nonzero for any* $\pi \in \Pi_\Psi(\mathrm{U}(V))$.

Proof. This is [Mœg11, Thm. 5.2]. □

Remark 2.2.

(1) There is a caveat here: Mœglin's result [Mœg11, Thm. 5.2] is for the symplectic-orthogonal dual pair. If one assumes Mœglin's explicit construction of A-packets for unitary groups (both quasi-split and nonquasi-split), then Mœglin's proof of [Mœg11, Thm. 5.2] should also work for unitary dual pairs.

(2) In later proofs of our main result, we will only use A-packets of unitary groups in some special cases; those A-packets are the Zelevinsky–Aubert dual of some tempered L-packets. Since the LLC for unitary groups has been fully established (see [Mok15, KMSW14, MR18, CZ21a]), all the properties of those A-packets that we need can be easily checked using the properties of the LLC and the Zelevinsky–Aubert duality. Hence, our main result in this paper is not conditional on the construction of A-packets for unitary groups.

Recall that for each local A-parameter Ψ, the local A-packet $\Pi_\Psi(\mathrm{U}(V))$ is also equipped with a map (depending on the choice of the additive character ψ_F)

$$\mathcal{J} : \Pi_\Psi(\mathrm{U}(V)) \longrightarrow \mathrm{Irr}\, A_\Psi,$$

where A_Ψ is the component group associated to Ψ. For example, if Ψ is a local A-parameter of good parity as above, then

$$A_\Psi = \sum_j \mathbb{Z}/2\mathbb{Z}\, a_j$$

is a free $\mathbb{Z}/2\mathbb{Z}$-module with a canonical basis $\{a_j\}_j$, where j runs over a representative set of inequivalent subrepresentations of Ψ.

Theorem 2.3. *In the context of Theorem 2.1(1), let* $\pi \in \Pi_\Psi(\mathrm{U}(V))$ *and* η *the character of* A_Ψ *associated to* π*. If the theta lift* $\theta(\pi)$ *is nonzero, then it corresponds to the character* $\theta(\eta)$ *of* $A_{\theta(\Psi)}$*, where* $\theta(\eta)$ *can be uniquely determined as follows:*

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- *if n and m are of different parity, then*

$$\theta(\eta)\Big|_{A_\Psi} = \eta;$$

- *if n and m are of the same parity, then*

$$\theta(\eta)(a_j)/\eta(a_j) = \epsilon\left(\frac{1}{2}, \Psi_j\chi_W^{-1}, \psi_{E,\delta}\right),$$

  where $a_j \in A_\Psi$ is the basis element corresponding to the irreducible summand Ψ_j of Ψ.

Here, we regard A_Ψ as a subgroup of $A_{\theta(\Psi)}$ via the canonical injection $A_\Psi \hookrightarrow A_{\theta(\Psi)}$ sending each basis element $a_j \in A_\Psi$ corresponding to $\Psi_j \subset \Psi$ to the basis element $a'_j \in A_{\theta(\Psi)}$ corresponding to $\Psi_j\chi_W^{-1}\chi_V \subset \theta(\Psi)$.

Proof. This can be proved as in [Ato18, Sect. 7.4]. See also [CZ21b, Cor. 7.4]. □

2.3. A result of Atobe

The following lemma, which is essentially due to Atobe, is useful to us in the later proofs. Let ν be the normalized absolute value of $E^\times$.

Lemma 2.4. *Let $G_0 = \mathrm{U}(V_0)$ be the unitary group associated to some Hermitian (or skew-Hermitian) space V_0, and Ψ_0 an A-parameter of G_0. Suppose that Ψ_0 is of good parity, multiplicity free and trivial on the Weil–Deligne $\mathrm{SL}_2(\mathbb{C})$. Let ρ be an irreducible representation of W_E and $x \in \frac{1}{2}\mathbb{Z}$ positive such that*

$$\rho\boxtimes S_{2x-1} \subset \Psi_0 \quad \text{and} \quad \rho\boxtimes S_{2x+1} \not\subset \Psi_0.$$

Then for any $\pi_0 \in \Pi_{\Psi_0}(G_0)$, we have a nonsplit exact sequence:

$$0 \longrightarrow \pi \longrightarrow \rho\nu^{-x} \rtimes \pi_0 \longrightarrow \pi' \longrightarrow 0,$$

where π is the unique irreducible subrepresentation and π' is the unique irreducible quotient of $\rho\nu^{-x} \rtimes \pi_0$. In particular, the length of the induced representation $\rho\nu^{-x} \rtimes \pi_0$ is 2.

Proof. Let $\phi_0 = \widehat{\Psi_0}$ be the Aubert dual of ψ_0, namely the L-parameter of G_0 obtained from Ψ_0 by exchanging the Weil–Deligne $\mathrm{SL}_2(\mathbb{C})$ and the Arthur $\mathrm{SL}_2(\mathbb{C})$. By our assumption, ϕ_0 is a discrete L-parameter. Then apply [Ato20, Lem. 5.1] to $\widehat{\pi_0} \in \Pi_{\phi_0}(G_0)$. □

Remark 2.5. Although Atobe only considered split odd orthogonal groups and symplectic groups in [Ato20], his Lemma 5.1 is true for unitary groups as well. Indeed the three ingredients used in his proof of orthogonal/symplectic groups are: Mœglin's construction of tempered L-packets, Tadić's formula and a lemma of Gan–Ichino [GI16, Lem. A.6]. Since all of these three ingredients are also valid for unitary groups, his proof also works for unitary groups with very minor modifications.

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2.4. Some local A-packets

Now, we use the Adams' conjecture to describe Weil representations. Let E^1 be the subgroup of $E^\times$ consists of norm 1 elements. Let χ_0 be a character of E^1 and χ the character of $E^\times$ obtained from χ_0 by base change; we may regard χ as the L-parameter of the unitary group E^1 corresponding to χ_0. We denote by $\omega_{V,\mu}[\chi]$ the maximal subrepresentation of $\omega_{V,\mu}$ such that the center of $\mathrm{U}(V)$ acts by χ_0. When $n=1$, the representation $\omega_{V,\mu}[\chi]$ has been studied by [Moe87] and [Rog92]. So we shall concentrate on the case $n \geq 2$.

Lemma 2.6.

(1) *If $n=2$ and $\chi=\mu^2$, then the representation $\omega_{V,\mu}[\chi]$ is nonzero only when the space V is of sign $+1$, in which case $\omega_{V,\mu}[\chi]$ is the generic member (with respect to the generic datum defined by ψ_F) in the L-packet $\Pi_\phi(\mathrm{U}(V))$, where*

$$\phi = \mu + \mu.$$

(2) *For any $n=\dim V \geq 2$, excluding the special case above, the representation $\omega_{V,\mu}[\chi]$ is nonzero, irreducible and unitary. It lies in the A-packet $\Pi_\Psi(\mathrm{U}(V))$, where*

$$\Psi = \chi \cdot \mu^{-n+1} + \mu \boxtimes S_{n-1}.$$

The character $\eta \in \mathrm{Irr}\, A_\Psi$ associated to $\omega_{V,\mu}[\chi]$ is

$$\eta : (e_1, e_{n-1}) \longmapsto \begin{cases} (1, \epsilon(V)) & \text{if } n \text{ is even,} \\ \left(\epsilon\left(\frac{1}{2}, \chi\mu^{-n}, \psi_{E,\delta}\right), \epsilon(V)\epsilon\left(\frac{1}{2}, \chi\mu^{-n}, \psi_{E,\delta}\right)\right) & \text{if } n \text{ is odd.} \end{cases}$$

Here, e_1 and e_{n-1} are the basis elements of A_Ψ corresponding to $\chi \cdot \mu^{-n+1}$ and $\mu \boxtimes S_{n-1}$, respectively.

Proof. Let L_1 be the one-dimensional Hermitian space associated to $1 \in F^\times$. Let χ_V be a character of $E^\times$ such that $\chi_V \big|_{F^\times} = \omega_{E/F}^n$ and $\Omega_{L_1,V}$ the Weil representation associated to $\mathrm{U}(L_1) \times \mathrm{U}(V)$ with respect to the splitting character (μ, χ_V). Then we have

$$\Omega_{L_1,V} \Big|_{\mathrm{U}(V)} = \omega_{V,\mu}.$$

Hence, $\omega_{V,\mu}[\chi]$ can be regarded as the theta lift of the character $\chi\mu^{-n}\chi_V$. Thus, our first assertion follows from Theorem 2.1, and the second follows from Theorem 2.3. $\square$

2.5. Irreducibility of big theta lifts

Finally, we investigate the irreducibility of the big theta lift of $\omega_{V,\mu}[\chi]$. We shall work in a slightly more general setting.

We retain the notations of Section 2.2. From now on, we assume that m is even and $m \geq \max\{2n-2, n\}$. Let

$$\Psi = \delta + \mu \boxtimes S_{n-1}$$

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be a local A-parameter of $\mathrm{U}(V)$, where δ and μ are conjugate self-dual characters of parity $(-1)^{n-1}$ and -1, respectively. Our goal is to show the following.

Theorem 2.7. *For any $\pi \in \Pi_\Psi(\mathrm{U}(V))$, the big theta lift $\Theta(\pi)$ to $\mathrm{U}(W)$ is irreducible if it is nonzero. Moreover, we have*

$$\mathrm{Ext}^i_{\mathrm{U}(V)}(\Omega,\pi)_{sm} = 0$$

for all $i > 0$. Here, Ω is the Weil representation associated to $\mathrm{U}(V)\times\mathrm{U}(W)$, and the subscript 'sm' stands for taking the $\mathrm{U}(W)$-smooth vectors.

Remark 2.8. Although in this theorem we do not assert the nonvanishing of $\Theta(\pi)$, in the range we are considering (i.e., $m \geq 2n-2$ and $n \geq 2$), we are almost always in the situation of Theorem 2.1(2), except for the following low rank cases:

- $n = 2$ and $m = 2$ (this case will not be used in the proof of our main theorem);
- $n = 3$, $m = 4$ and $\delta = \chi_W$.

We shall prove this theorem by induction on the dimension of V. Let $x_n = -n/2+1$. According to Mœglin [Mœg06, Sect. 2.4], we know that:

Lemma 2.9. *Assume that $n \geq 3$, and let $\pi \in \Pi_\Psi(\mathrm{U}(V))$. If π is not supercuspidal, then there exists a unique $\pi_0 \in \Pi_{\Psi_0}(\mathrm{U}(V_0))$ such that*

$$\pi \hookrightarrow \mu\nu^{x_n} \rtimes \pi_0.$$

Here,

$$\Psi_0 = \delta + \mu \boxtimes S_{n-3},$$

and V_0 is a subspace of V such that $V \simeq V_0 \oplus \mathcal{H}$.

Using this fact, we now do the induction step.

Proposition 2.10. *In the context of Lemma 2.9, assume that*

$$\pi \hookrightarrow \mu\nu^{x_n} \rtimes \pi_0.$$

Let W_0 be a subspace of W such that $W \simeq W_0 \oplus \mathcal{H}'$. Consider the theta correspondence of $\mathrm{U}(V_0)\times\mathrm{U}(W_0)$ (with respect to the same splitting characters). Then if Theorem 2.7 holds for π_0, it also holds for π.

Remark 2.11. In the setting of Theorem 2.7, we have assumed that $m \geq 2n-2$. Note that the dimensions of V_0 and W_0 also satisfy this inequality. Hence, it makes sense to talk about Theorem 2.7 for π_0.

Proof of Proposition 2.10. Let P be the standard parabolic subgroup of $\mathrm{U}(V)$ with Levi component $\mathrm{GL}_1 \times \mathrm{U}(V_0)$. For $i \geq 0$, by the (derived version of) Frobenius reciprocity, we have

$$\mathrm{Ext}^i_{\mathrm{U}(V)}(\Omega,\mu\nu^{x_n} \rtimes \pi_0) = \mathrm{Ext}^i_{\mathrm{GL}_1\times\mathrm{U}(V_0)}(R_P\Omega,\mu\nu^{x_n}\boxtimes\pi_0),$$

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where R_P is the normalized Jacquet module along P. To compute the right-hand side of above, one can appeal to the Kudla's filtration. There is a two-step filtration on $R_P\Omega$:

$$R_P\Omega = R^0 \supset R^1 \supset R^2 = 0,$$

whose successive quotient $J^a = R^a/R^{a+1}$ can be described as follows:

$$J^0 = \chi_W \nu^{\frac{m-n+1}{2}} \boxtimes \Omega_0,$$

and

$$J^1 = \mathrm{Ind}_{\mathrm{GL}_1\times \mathrm{U}(V_0)\times Q}^{\mathrm{GL}_1\times \mathrm{U}(V_0)\times \mathrm{U}(W)} \left(\mathcal{S}(E^\times)\boxtimes \Omega_{00}\right).$$

Here:

- Ω_0 is the Weil representation associated to $\mathrm{U}(V_0)\times \mathrm{U}(W)$;
- Q is a maximal parabolic subgroup of $\mathrm{U}(W)$ stabilizing an isotropic line of W; the Levi subgroup of Q is isomorphic to $\mathrm{GL}_1\times \mathrm{U}(W_0)$;
- $\mathcal{S}(E^\times)$ is the space of Schwartz functions on $E^\times$, equipped with the natural action of two copies of GL_1 (twisted by the splitting characters);
- Ω_{00} is the Weil representation associated to $\mathrm{U}(V_0)\times \mathrm{U}(W_0)$.

Applying the functor $\mathrm{Hom}_{\mathrm{GL}_1\times \mathrm{U}(V_0)}(\cdot, \mu\nu^{x_n}\boxtimes\pi_0)$ to the short exact sequence

$$0 \longrightarrow J^1 \longrightarrow R_P\Omega \longrightarrow J^0 \longrightarrow 0,$$

we get a long exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}^i(J^0,\mu\nu^{x_n}\boxtimes\pi_0)_{sm} \longrightarrow \mathrm{Ext}^i(R_P\Omega,\mu\nu^{x_n}\boxtimes\pi_0)_{sm}\\ \longrightarrow \mathrm{Ext}^i(J^1,\mu\nu^{x_n}\boxtimes\pi_0)_{sm} \longrightarrow \mathrm{Ext}^{i+1}(J^0,\mu\nu^{x_n}\boxtimes\pi_0)_{sm} \longrightarrow \cdots\end{aligned}$$

Here, for the exactness of taking $\mathrm{U}(W)$-smooth vectors, one may refer to [APS17, Lem. 5.14, Lem. 7.4]. Since by our assumptions $x_n \neq \frac{m-n+1}{2}$, we know that

$$\mathrm{Ext}^i(J^0,\mu\nu^{x_n}\boxtimes\pi_0) = 0$$

for all $i \geq 0$. This implies that

$$\begin{aligned}\mathrm{Ext}^i(R_P\Omega,\mu\nu^{x_n}\boxtimes\pi_0)_{sm} &\simeq \mathrm{Ext}^i(J^1,\mu\nu^{x_n}\boxtimes\pi_0)_{sm}\\ &= (\mu')^c\nu^{x_n}\rtimes \mathrm{Ext}^i(\Omega_{00},\pi_0)_{sm},\end{aligned}$$

where $\mu' = \mu\chi_W^{-1}\chi_V$. Here, in the last equality, we have made use of (an Ext-version of) [APS17, Lem. 2.6] and the Künneth formula [APS17, Lem. 3.3]. In particular, we get

$$\mathrm{Hom}_{\mathrm{U}(V)}(\Omega,\mu\nu^{x_n}\rtimes\pi_0)_{sm} \simeq (\mu')^c\nu^{x_n}\rtimes\Theta(\pi_0)^\vee$$

and

$$\mathrm{Ext}^i_{\mathrm{U}(V)}(\Omega,\mu\nu^{x_n}\rtimes\pi_0)_{sm} = 0 \quad \textit{for } i>0 \tag{$\spadesuit$}$$

by our induction hypothesis. Similarly, we also have $-x_n \neq \frac{m-n+1}{2}$. The same argument gives that

$$\mathrm{Ext}^i_{\mathrm{U}(V)}(\Omega,\mu\nu^{-x_n}\rtimes\pi_0)_{sm} = 0 \quad \textit{for } i>0. \tag{$\clubsuit$}$$

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Now, note that Lemma 2.4 asserts that $\mu\nu^{x_n} \rtimes \pi_0$ is of length 2. Let π' be the unique irreducible quotient of $\mu\nu^{x_n} \rtimes \pi_0$, the sequence

$$0 \longrightarrow \pi \longrightarrow \mu\nu^{x_n} \rtimes \pi_0 \longrightarrow \pi' \longrightarrow 0$$

is exact. Applying the functor $\mathrm{Hom}_{\mathrm{U}(V)}(\Omega, \cdot)$ to this short exact sequence and taking $\mathrm{U}(W)$-smooth vectors, we get

$$\begin{aligned}&0 \to \mathrm{Hom}(\Omega,\pi)_{sm} \to \mathrm{Hom}(\Omega,\mu\nu^{x_n} \rtimes \pi_0)_{sm} \to \mathrm{Hom}(\Omega,\pi')_{sm} \to \mathrm{Ext}^1(\Omega,\pi)_{sm} \to \\ &\cdots \to \mathrm{Ext}^i(\Omega,\mu\nu^{x_n} \rtimes \pi_0)_{sm} \to \mathrm{Ext}^i(\Omega,\pi')_{sm} \to \mathrm{Ext}^{i+1}(\Omega,\pi)_{sm} \to \mathrm{Ext}^{i+1}(\Omega,\mu\nu^{x_n} \rtimes \pi_0)_{sm} \to \cdots\end{aligned} \tag{$\heartsuit$}$$

It follows from Equation ($\spadesuit$) that

$$\mathrm{Ext}^i(\Omega,\pi')_{sm} \simeq \mathrm{Ext}^{i+1}(\Omega,\pi)_{sm} \quad \textit{for } i > 0.$$

Dually, apply both the contragredient and MVW-involution (see [MVW, Chap. 4.II.1]) to $\mu\nu^{x_n} \rtimes \pi_0$, we get a dualized short exact sequence

$$0 \longrightarrow \pi' \longrightarrow \mu\nu^{-x_n} \rtimes \pi_0 \longrightarrow \pi \longrightarrow 0.$$

Similar to the argument above, this short exact sequence leads to a long exact sequence

$$\begin{aligned}&0 \to \mathrm{Hom}(\Omega,\pi')_{sm} \to \mathrm{Hom}(\Omega,\mu\nu^{-x_n} \rtimes \pi_0)_{sm} \to \mathrm{Hom}(\Omega,\pi)_{sm} \to \mathrm{Ext}^1(\Omega,\pi')_{sm} \to \\ &\cdots \to \mathrm{Ext}^i(\Omega,\mu\nu^{-x_n} \rtimes \pi_0)_{sm} \to \mathrm{Ext}^i(\Omega,\pi)_{sm} \to \mathrm{Ext}^{i+1}(\Omega,\pi')_{sm} \to \mathrm{Ext}^{i+1}(\Omega,\mu\nu^{-x_n} \rtimes \pi_0)_{sm} \to \cdots\end{aligned} \tag{$\diamondsuit$}$$

which combining with Equation ($\clubsuit$) similarly implies that

$$\mathrm{Ext}^i(\Omega,\pi)_{sm} \simeq \mathrm{Ext}^{i+1}(\Omega,\pi')_{sm} \quad \textit{for } i > 0.$$

Playing ‘Ping-Pong’, one can see that $\mathrm{Ext}^i(\Omega,\pi)$ is periodic:

$$\mathrm{Ext}^i(\Omega,\pi)_{sm} \simeq \mathrm{Ext}^{i+2}(\Omega,\pi)_{sm} \quad \textit{for } i > 0.$$

Since the higher extensions vanish when the degree is sufficiently large [Ber92, Pg. 98, Sect. 4.2], these groups $\mathrm{Ext}^i(\Omega,\pi)_{sm}$ must vanish for all $i > 0$ with no other choice. The same reason also gives the vanishing of higher extensions of π'.

Suppose that $\Theta(\pi) \neq 0$. It remains to show that $\Theta(\pi)$ is irreducible. Thanks to the vanishing of higher extensions, we deduce from the long exact sequence ($\heartsuit$) that

$$0 \longrightarrow \Theta(\pi)^\vee \longrightarrow (\mu')^c\nu^{x_n} \rtimes \Theta(\pi_0)^\vee \longrightarrow \Theta(\pi')^\vee \longrightarrow 0$$

is exact. In particular, $\Theta(\pi_0)$ must be nonzero, hence irreducible by our induction hypothesis. It then follows from Theorem 2.1 that $\Theta(\pi_0)$ lies in $\Pi_{\theta(\Psi_0)}(\mathrm{U}(W_0))$, where

$$\theta(\Psi_0) = \delta\chi_W^{-1}\chi_V + \mu\chi_W^{-1}\chi_V \boxtimes S_{n-3} + \chi_V \boxtimes S_{m-n}.$$

Now, we claim that the induced representation $(\mu')^c\nu^{x_n} \rtimes \Theta(\pi_0)^\vee$ is of length 2, and the two subquotients are nonisomorphic. Indeed, if $m-n=1$ and $\delta = \chi_W$, one can easily check this by hand. Otherwise, note that:

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- $\theta(\Psi_0)$ is of good parity, multiplicity free and trivial on the Weil–Deligne $\mathrm{SL}_2(\mathbb{C})$;
- $\mu' \boxtimes S_{-2x_n-1} \subset \theta(\Psi_0)$ but $\mu' \boxtimes S_{-2x_n+1} \not\subset \theta(\Psi_0)$.

In short, we are again in a situation such that we can appeal to Lemma 2.4, from which our claim follows. Therefore, it suffices to check that $\Theta(\pi') \neq 0$. We shall argue by contradiction to show this. Suppose on the contrary that $\Theta(\pi') = 0$. Then on the one hand, we have

$$\Theta(\pi)^\vee \simeq (\mu')^c \nu^{x_n} \rtimes \Theta(\pi_0)^\vee,$$

which implies that $(\mu')^c \nu^{x_n} \rtimes \Theta(\pi_0)^\vee$ has socle $\theta(\pi)^\vee$. On the other hand, we also deduce from the long exact sequence ($\diamondsuit$) that

$$0 \longrightarrow \Theta(\pi')^\vee \longrightarrow (\mu')^c \nu^{-x_n} \rtimes \Theta(\pi_0)^\vee \longrightarrow \Theta(\pi)^\vee \longrightarrow 0$$

is exact. Since we had assumed that $\Theta(\pi') = 0$, this exact sequence implies that

$$(\mu')^c \nu^{-x_n} \rtimes \Theta(\pi_0)^\vee \simeq \Theta(\pi)^\vee.$$

Applying both the contragredient and the MVW-involution, we get

$$\Theta(\pi)^{MVW} \simeq (\mu')^c \nu^{x_n} \rtimes \Theta(\pi_0)^\vee,$$

which implies that $(\mu')^c \nu^{x_n} \rtimes \Theta(\pi_0)^\vee$ also has cosocle $\theta(\pi)^\vee$. This contradicts our claim. Thus, $\Theta(\pi') \neq 0$ as desired. □

Now, we can prove our goal.

Proof of Theorem 2.7. By using the previous proposition, we can reduce Theorem 2.7 to the case that π is supercuspidal or to the case that $n = 0$. In the supercuspidal case:

- it is well known that the big theta lift of a supercuspidal representation is irreducible;
- all higher extensions vanish since supercuspidal representations of a unitary group are compact.

In the case that $n = 0$, $\mathrm{U}(V)$ is trivial and the Weil representation is simply a character of $\mathrm{U}(W)$. Hence, Theorem 2.7 holds. □

3. Proof of the main result

In this section, we shall prove the main result: Theorem 1.3. We first note:

Lemma 3.1. *Assume that Conjecture 1.1 holds for a tempered L-parameter M. Then for any conjugate orthogonal character $\mathcal{X}$ of $L^\times$, Conjecture 1.1 also holds for the L-parameter $M \cdot \mathcal{X}$.*

Proof. To see this, one simply notes that

$$m_V(\Pi, \mu) = m_V\Big(\Pi \cdot \mathcal{X}_0, \mu\left(\mathcal{X} \mid_{E^\times}\right)\Big),$$

where $\mathcal{X}_0$ is the character of L^1 whose base change to $L^\times$ is $\mathcal{X}$. □

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Let $n \geq 2$ be an integer, and V an $(n+1)$-dimensional skew-Hermitian space over E. We shall start with an L-parameter of the form

$$M = M_0 + M_1,$$

where M_1 is a conjugate self-dual character of parity $(-1)^n$.

3.1. Two seesaw diagrams: uniqueness

If there is an irreducible tempered representation Π in the L-packet Π_M corresponding to $\eta \in \mathrm{Irr}\, A_M$ such that

$$m_V(\Pi, \mu) \neq 0,$$

we would like to lift Π to some unitary group of n-variables to obtain some information. Let $\{a_i\}_{i=1}^r$ be a canonical basis of A_M, where each a_i corresponds to some irreducible subrepresentation M_i of M (so a_1 corresponds to M_1). We set $\epsilon = \eta(a_1)$ and W the unique n-dimensional Hermitian space over L of sign ϵ. Let $(\mathcal{X}_V, \mathcal{X}_W)$ be a pair of characters of $L^\times$ such that

$$\mathcal{X}_V \mid_{K^\times} = \omega_{L/K}^{n+1} \quad \textit{and} \quad \mathcal{X}_W = M_1.$$

Then one can consider the theta correspondence between $\mathrm{U}(V_K) \times \mathrm{U}(W)$ with respect to the splitting character $(\mathcal{X}_V, \mathcal{X}_W)$. By [GI16, Sect. 4.6(P2)], one knows that there is an irreducible tempered representation Σ of $\mathrm{U}(W)$ such that

$$\Pi = \Theta(\Sigma)$$

is the big theta lift of Σ. Indeed, one knows that Σ has the L-parameter

$$\theta(M) = M_0 \cdot \mathcal{X}_W^{-1} \mathcal{X}_V$$

and corresponds to the character $\theta(\eta) = \eta \mid_{A_{\theta(M)}}$. Consider the following seesaw diagram:

$$\begin{tikzcd}
\mathrm{U}(\mathcal{R}W) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{U}(V_K) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{U}(W) & & \mathrm{U}(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda = \Theta(\omega_{V,\mu}[\chi]) \arrow[d, dash] \arrow[drr, dash] & & \Pi = \Theta(\Sigma) \arrow[d, dash] \arrow[dll, dash] \\
\Sigma & & \omega_{V,\mu}[\chi]
\end{tikzcd}
\tag{$\natural$.1}$$

where:

- $\mathcal{R}W$ is the restriction of scalar of W to E;
- the theta correspondence between $\mathrm{U}(V_K) \times \mathrm{U}(W)$ is with respect to some splitting characters $(\mathcal{X}_V, \mathcal{X}_W)$;
- the theta correspondence between $\mathrm{U}(V) \times \mathrm{U}(\mathcal{R}W)$ is with respect to some splitting characters (χ_V, χ_W);

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- to make use of this seesaw diagram, we choose these splitting characters so that:

$$\mathcal{X}_V = \chi_V \circ \mathrm{Nm}_{L/E} \quad \textit{and} \quad \chi_W = \mathcal{X}_W \big|_{E^\times};$$

- χ is the L-parameter of the central character of the restriction of Π to $\mathrm{U}(V)$, that is,

$$\chi = \det(M) \big|_{E^\times}.$$

Then by the seesaw identity, we get

$$m_V(\Pi,\mu) = \dim \mathrm{Hom}_{\mathrm{U}(W)}(\Lambda,\Sigma). \tag{✠.1}$$

In particular, Λ is nonzero. By Lemma 2.6, Theorem 2.1 and Theorem 2.7, we know that:

- $\omega_{V,\mu}[\chi]$ lies in the A-packet $\Pi_{\Psi_{M,\mu}}(\mathrm{U}(V))$, where

$$\Psi_{M,\mu} = \chi\cdot\mu^{-n} + \mu \boxtimes S_n;$$

- Λ is an irreducible unitary representation lies in the A-packet $\Pi_{\theta(\Psi_{M,\mu})}(\mathrm{U}(\mathcal{R}W))$, where

$$\begin{aligned}\theta(\Psi_{M,\mu}) &= \Psi_{M,\mu}\cdot\chi_W^{-1}\chi_V + \chi_V \boxtimes S_{n-1}\\ &= \chi\cdot\mu^{-n}\cdot\chi_W^{-1}\chi_V + \chi_V\boxtimes S_{n-1} + \mu\cdot\chi_W^{-1}\chi_V\boxtimes S_n.\end{aligned}$$

To compute the right-hand side of equality (✠.1), we shall use another seesaw diagram:

$$\begin{tikzcd}
\mathrm{U}\left(V_K^\flat\right) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{U}(\mathcal{R}W) \arrow[d, dash] \arrow[dll, dash] & & \Pi^\flat = \Theta(\Sigma) \arrow[d, dash] \arrow[drr, dash] & & \Theta(\omega) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{U}(V^\flat) & & \mathrm{U}(W) & & \omega & & \Sigma
\end{tikzcd} \tag{♮.2}$$

where:

- $V^\flat$ is an n-dimensional skew-Hermitian space over E which will be suitably chosen later, and $V_K^\flat$ is its scalar extension to L;
- the theta correspondence between $\mathrm{U}(V^\flat)\times\mathrm{U}(\mathcal{R}W)$ is with respect to some splitting characters $(\chi_{V^\flat},\chi'_W)$;
- the theta correspondence between $\mathrm{U}(V_K^\flat)\times\mathrm{U}(W)$ is with respect to some splitting characters $(\mathcal{X}_{V^\flat},\mathcal{X}'_W)$;
- to make use of this seesaw diagram, we choose these splitting characters so that:

$$\mathcal{X}_{V^\flat} = \chi_{V^\flat}\circ\mathrm{Nm}_{L/E} \quad \textit{and} \quad \chi'_W = \mathcal{X}'_W \big|_{E^\times};$$

- ω is some irreducible unitary representation of $\mathrm{U}(V^\flat)$ which will also be suitably chosen later.

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We would like to choose these data appropriately such that ω is the theta lift of certain character of U_1, and $\Lambda = \Theta(\omega)$. To make this possible, we need to pick up these splitting characters very carefully. Let

$$\chi_{V^\flat} = \mu \cdot \chi_W^{-1}\chi_V \quad \text{and} \quad \mathcal{X}'_W = M_1^{-1} \cdot \Upsilon,$$

where Υ is a conjugate orthogonal character of $L^\times$ so that

$$\Upsilon \Big|_{E^\times} = \mu^2.$$

It is not hard to see that such Υ exists. Then again by Theorem 2.1 one can see that ω (if exists) lies in the A-packet $\Pi_{\Psi^\flat}(\mathrm{U}(V^\flat))$, where

$$\Psi^\flat = \chi^\flat \cdot \mu^{-n+1} + \mu \boxtimes S_{n-1}, \quad \text{with } \chi^\flat = \det(M/M_1) \Big|_{E^\times}.$$

Indeed, we have:

Proposition 3.2. *Let $V^\flat$ be the n-dimensional skew-Hermitian space of sign*

$$\epsilon\left(V^\flat\right) = \begin{cases} +1 & \text{if } n = 2 \text{ and } \chi^\flat = \mu^2, \\ \epsilon(V) \cdot \epsilon(\mathcal{R}W) \cdot \epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}(M_1) \cdot \mu^{-1}, \psi_{E,\delta}\right) \cdot \omega_{E/F}(-1)^n & \text{otherwise,} \end{cases} \tag{†}$$

and

$$\omega = \omega_{V^\flat,\mu}[\chi^\flat].$$

Then Λ is the (big) theta lift of ω to $\mathrm{U}(\mathcal{R}W)$, that is, $\Lambda = \Theta(\omega)$.

Proof. We first check the special case that $n = 2$ and $\chi^\flat = \mu^2$. So

$$\theta(\Psi_{M,\mu}) = \chi_V + \mu \cdot \chi_W^{-1}\chi_V \boxtimes S_2 + (\chi_V^c)^\vee$$

and

$$\Lambda \subset \chi_V \rtimes (\chi_0 \circ \det),$$

where χ_0 is the character of E^1 whose base change to $E^\times$ is $\mu \cdot \chi_W^{-1}\chi_V$. By the induction principle, one knows that the theta correspondence between $\mathrm{U}(V^\flat) \times \mathrm{U}(\mathcal{R}W)$ defines a bijection

$$\theta : \Pi_\phi(\mathrm{U}(V^\flat)) \longrightarrow \Pi_{\theta(\Psi_{M,\mu})}(\mathrm{U}(\mathcal{R}W)),$$

where $\phi = \mu + \mu$ is an L-parameter of $\mathrm{U}(V^\flat)$. Hence, Λ is the (big) theta lift of some $\omega \in \Pi_\phi(\mathrm{U}(V^\flat))$. To check that $\omega = \omega_{V^\flat,\mu}[\chi^\flat]$, one can compute the character $\eta^\flat \in \mathrm{Irr}\, A_\phi$ associated to ω. Recall that Λ is also the theta lift of $\omega_{V,\mu}[\chi]$. If we denote by $\eta \in \mathrm{Irr}\, A_{\Psi_{M,\mu}}$ and $\theta(\eta) \in \mathrm{Irr}\, A_{\theta(\Psi_{M,\mu})}$ the character associated to $\omega_{V,\mu}[\chi]$ and Λ, respectively, then by Lemma 2.6 and Theorem 2.3, we have

$$\theta(\eta)(a) = \eta(a) = \epsilon\left(\frac{1}{2}, \chi_W \mu^{-1}, \psi_{E,\delta}\right).$$

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Here, $a \in A_{\theta(\Psi_{M,\mu})}$ is the basis element corresponding to χ_V, and we regard A_ϕ and $A_{\Psi_{M,\mu}}$ as subgroups of $A_{\theta(\Psi_{M,\mu})}$. Apply Theorem 2.3 again, we get

$$\eta^\flat(a) = \theta(\eta)(a) \cdot \epsilon\left(\frac{1}{2}, \chi_V \cdot \chi_{V^\flat}^{-1}, \psi_{E,\delta}\right) = 1.$$

This implies that $\omega = \omega_{V^\flat,\mu}[\chi^\flat]$.

Now, excluding the special case above, we prove the general case. It would be convenient to consider the cases of odd and even n separately. In the following, we check the case of odd n in full details.

Let e_1, e_{n-1} and e_n be the basis elements of $A_{\theta(\Psi_{M,\mu})}$ corresponding to $\chi \cdot \mu^{-n} \cdot \chi_W^{-1}\chi_V$, $\chi_V \boxtimes S_{n-1}$ and $\mu \cdot \chi_W^{-1}\chi_V \boxtimes S_n$, respectively. Then:

- $A_{\Psi_{M,\mu}}$ can be regarded as the subgroup of $A_{\theta(\Psi_{M,\mu})}$ generated by e_1 and e_n;
- $A_{\Psi^\flat}$ can be regarded as the subgroup of $A_{\theta(\Psi_{M,\mu})}$ generated by e_1 and e_{n-1}.

Recall that $\omega_{V,\mu}[\chi] \in \Pi_{\Psi_{M,\mu}}(\mathrm{U}(V))$ corresponds to the character ν_{n+1} of $A_{\Psi_{M,\mu}}$ such that

$$\nu_{n+1} : (e_1, e_n) \longmapsto (1, \epsilon(V)).$$

Then by Theorem 2.3, $\Lambda = \Theta(\omega_{V,\mu}[\chi])$ corresponds to the character ν of $A_{\theta(\Psi_{M,\mu})}$ such that

$$\nu : (e_1, e_n) \longmapsto \left(\epsilon\left(\frac{1}{2}, \chi \cdot \mu^{-n} \cdot \chi_W^{-1}, \psi_{E,\delta}\right), \epsilon(V) \cdot \epsilon\left(\frac{1}{2}, \mu \cdot \chi_W^{-1} \boxtimes S_n, \psi_{E,\delta}\right)\right).$$

The evaluation of ν at e_{n-1} can be determined by its evaluation at (e_1, e_n) and the sign of $\mathcal{R}W$. To be more precise, ν takes e_{n-1} to

$$\begin{aligned} &\epsilon(V) \cdot \epsilon(\mathcal{R}W) \cdot \epsilon\left(\frac{1}{2}, \chi \cdot \mu^{-n} \cdot \chi_W^{-1}, \psi_{E,\delta}\right) \cdot \epsilon\left(\frac{1}{2}, \mu \cdot \chi_W^{-1} \boxtimes S_n, \psi_{E,\delta}\right) \\ = &\epsilon(V) \cdot \epsilon(\mathcal{R}W) \cdot \epsilon\left(\frac{1}{2}, \chi^\flat \cdot \mu^{-n}, \psi_{E,\delta}\right) \cdot \epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}(M_1) \cdot \mu^{-1}, \psi_{E,\delta}\right) \cdot \omega_{E/F}(-1). \end{aligned}$$

Hence, if we let $V^\flat$ be the n-dimensional skew-Hermitian space as in Equation (†), then again by Theorem 2.3, one can check that:

- $\omega_{V^\flat,\mu}[\chi^\flat] \in \Pi_{\Psi^\flat}(\mathrm{U}(V^\flat))$ corresponding to the character ν_n of $A_{\Psi^\flat}$ such that
$$\nu_n : (e_1, e_{n-1}) \longmapsto \left(\epsilon\left(\frac{1}{2}, \chi^\flat \cdot \mu^{-n}, \psi_{E,\delta}\right), \epsilon\left(V^\flat\right) \cdot \epsilon\left(\frac{1}{2}, \chi^\flat \cdot \mu^{-n}, \psi_{E,\delta}\right)\right);$$
- the theta lift of $\omega_{V^\flat,\mu}[\chi^\flat]$ to $\mathrm{U}(\mathcal{R}W)$ is nonzero and exactly equal to Λ.

These complete the proof of the case when n is odd.

Similarly, when n is even, $\omega_{V,\mu}[\chi] \in \Pi_{\Psi_{M,\mu}}(\mathrm{U}(V))$ corresponds to

$$\nu_{n+1} : (e_1, e_n) \longmapsto \left(\epsilon\left(\frac{1}{2}, \chi\mu^{-n-1}, \psi_{E,\delta}\right), \epsilon(V)\epsilon\left(\frac{1}{2}, \chi\mu^{-n-1}, \psi_{E,\delta}\right)\right).$$

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By Theorem 2.3, Λ corresponds to $\nu \in \operatorname{Irr} A_{\theta(\Psi_{M,\mu})}$ such that $\nu \big|_{A_{\Psi_{M,\mu}}} = \nu_{n+1}$, so

$$\nu(e_{n-1}) = \epsilon(V) \cdot \epsilon(\mathcal{R}W).$$

Then again one can appeal to Theorem 2.3 to show that the theta lift of $\omega_{V^\flat,\mu}[\chi^\flat]$ is exactly Λ. □

With this proposition in hand, we get

$$m_V(\Pi,\mu) = \dim \operatorname{Hom}_{\mathrm{U}(W)}(\Lambda,\Sigma) = m_{V^\flat}(\Pi^\flat,\mu) \tag{✠.2}$$

is nonzero. In particular, $\Pi^\flat$ is nonzero. By [GI16, Sect. 4.4(P1)], we know that:

- The sign of the Hermitian space W is given by

$$\epsilon(W) = \epsilon\left(\frac{1}{2}, M_0 \cdot {}^\tau M_1 \cdot \mu^{-1} \circ \mathrm{Nm}_{L/E}, \psi_{L,\delta}\right),$$

  where $\psi_{L,\delta} = \psi_F\left(\mathrm{Tr}_{L/F}(\delta \cdot \ )\right)$.
- $\Pi^\flat$ is an irreducible tempered representation has L-parameter $M^\flat = M_0'$ and corresponds to $\eta^\flat$, where

$$M_0' = M_0 \cdot {}^\tau M_1 \cdot M_1^{-1} \cdot \Upsilon \cdot \mu^{-1} \circ \mathrm{Nm}_{L/E},$$

  and

$$\eta^\flat(a_i)/\eta(a_i) = \epsilon\left(\frac{1}{2}, M_i \cdot {}^\tau M_1 \cdot \mu^{-1} \circ \mathrm{Nm}_{L/E}, \psi_{L,\delta}\right)$$

  for all $i \geq 2$.

Also note that

$$\epsilon(\mathcal{R}W) = \epsilon(W) \cdot \omega_{K/F}\left(\delta^2\right)^n \cdot \omega_{E/F}(-1)^n.$$

Substitute these into Equation (†), we get

$$\epsilon\left(V^\flat\right) = \epsilon(V) \cdot \epsilon\left(\frac{1}{2}, \mathrm{Ind}_L^E\left({}^\tau M_1 \otimes (M/M_1)\right) \cdot \mu^{-1}, \psi_{E,\delta}\right) \\ \cdot \epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}(M_1) \cdot \mu^{-1}, \psi_{E,\delta}\right) \cdot \omega_{K/F}\left(\delta^2\right)^n. \tag{††}$$

Now, if we assume that Conjecture 1.1 holds for the L-parameter $M^\flat$, then it follows that:

(1) The multiplicity $m_V(\Pi,\mu) = 1$.

(2) V is the unique $(n+1)$-dimensional Hermitian space over E predicted by the formula in Conjecture 1.1(3). Indeed, note that for any semisimple representation N and any character $\mathcal{X}$ of WD_L, we have

$$\mathrm{As}^+(N \cdot \mathcal{X}) = \mathrm{As}^+(N) \cdot \left(\mathcal{X}\big|_{E^\times}\right).$$

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Combining this with Conjecture 1.1(3) for $M^\flat$, we know that

$$\begin{aligned}\epsilon\left(V^\flat\right) &= \epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}\left(M^\flat\right)\otimes\mu^{-1},\psi_{E,\delta}\right)\cdot\omega_{K/F}\left(\delta^2\right)^{n(n-1)/2}\\ &= \epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}\left(M_0\right)\otimes\mu^{-1},\psi_{E,\delta}\right)\cdot\omega_{K/F}\left(\delta^2\right)^{n(n-1)/2}.\end{aligned}$$

Then applying the equality (††), we get

$$\begin{aligned}\epsilon(V) = \epsilon\left(V^\flat\right)\cdot\epsilon&\left(\frac{1}{2}, \mathrm{Ind}^E_L\left({}^\tau M_1\otimes(M/M_1)\right)\cdot\mu^{-1},\psi_{E,\delta}\right)\\ &\cdot\epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}(M_1)\cdot\mu^{-1},\psi_{E,\delta}\right)\cdot\omega_{K/F}\left(\delta^2\right)^n\\ = \epsilon&\left(\frac{1}{2}, \mathrm{As}^+_{L/E}(M)\otimes\mu^{-1},\psi_{E,\delta}\right)\cdot\omega_{K/F}\left(\delta^2\right)^{n(n+1)/2}.\end{aligned}$$

(3) Π is the unique member in Π_M predicted by the formula in Conjecture 1.1(4). Similar to (2), it follows from Conjecture 1.1(4) that

$$\begin{aligned}\eta^\flat(a_i) &= \epsilon\left(\frac{1}{2}, \mathrm{Ind}^E_L\left({}^\tau M_i\otimes(M_0/M_i)\right)\cdot\mu^{-1},\psi_{E,\delta}\right)\\ &= \epsilon\left(\frac{1}{2}, {}^\tau M_i\otimes(M_0/M_i)\cdot\mu^{-1}\circ\mathrm{Nm}_{L/E},\psi_{L,\delta}\right)\end{aligned}$$

for all $i\geq 2$. Hence,

$$\begin{aligned}\eta(a_i) = \eta^\flat(a_i)\cdot\epsilon&\left(\frac{1}{2}, M_i\cdot{}^\tau M_1\cdot\mu^{-1}\circ\mathrm{Nm}_{L/E},\psi_{L,\delta}\right)\\ &= \epsilon\left(\frac{1}{2}, {}^\tau M_i\otimes(M/M_i)\cdot\mu^{-1}\circ\mathrm{Nm}_{L/E},\psi_{L,\delta}\right)\\ &= \epsilon\left(\frac{1}{2}, \mathrm{Ind}^E_L\left({}^\tau M_i\otimes(M/M_i)\right)\cdot\mu^{-1},\psi_{E,\delta}\right)\end{aligned}$$

for all $i\geq 2$. On the other hand, recall that $\eta(a_1)=\epsilon(W)$. This implies the desired equality

$$\begin{aligned}\eta(a_1) &= \epsilon\left(\frac{1}{2}, M_0\cdot{}^\tau M_1\cdot\mu^{-1}\circ\mathrm{Nm}_{L/E},\psi_{L,\delta}\right)\\ &= \epsilon\left(\frac{1}{2}, \mathrm{Ind}^E_L\left({}^\tau M_1\otimes(M/M_1)\right)\cdot\mu^{-1},\psi_{E,\delta}\right).\end{aligned}$$

The computation above shows that there is at most one Π in the L-packet Π_M such that $m_V(\Pi,\mu)\neq 0$.

3.2. Reversing two seesaw diagrams: existence

Conversely, still under the assumption that Conjecture 1.1 holds for the L-parameter $M^\flat$, we can produce an irreducible tempered representation $\Pi' \in \Pi_M$ such that

$$m_{V'}(\Pi', \mu) \neq 0$$

for some $(n+1)$-dimensional skew-Hermitian space V', from the unique irreducible tempered representation $\Pi^\flat \in \Pi_{M^\flat}$ such that

$$m_{V^\flat}(\Pi^\flat, \mu) \neq 0.$$

We do it by applying the two seesaw diagrams reversely as follows. First, consider an analog of the seesaw diagram (♮.2) (using the same splitting characters):

$$\begin{array}{ccc} \mathrm{U}\left(V_K^\flat\right) & & \mathrm{U}\left(\mathcal{R}W'\right) \\ \Big| & \times & \Big| \\ \mathrm{U}(V^\flat) & & \mathrm{U}\left(W'\right) \end{array} \qquad \begin{array}{ccc} \Pi^\flat = \Theta(\Sigma') & & \Lambda' = \Theta\left(\omega_{V^\flat,\mu}[\chi^\flat]\right) \\ \Big| & \times & \Big| \\ \omega_{V^\flat,\mu}[\chi^\flat] & & \Sigma' \end{array},$$

where W' is the unique n-dimensional Hermitian space over L chosen by the theta dichotomy [GI16, Sect. 4.4(P1)]; that is, the theta lift Σ' of $\Pi^\flat$ to $\mathrm{U}(W')$ is nonzero. Symmetrically, $\Pi^\flat = \Theta(\Sigma')$ is the big theta lift of Σ'. By the seesaw identity, we have

$$m_{V^\flat}\left(\Pi^\flat, \mu\right) = \dim \mathrm{Hom}_{\mathrm{U}(W')}(\Lambda', \Sigma'). \tag{✠.3}$$

In particular, Λ' is nonzero. It then follows from Theorem 2.1 and Theorem 2.3 that Λ' is an irreducible unitary representation lies in the A-packet $\Pi_{\theta(\Psi_{M,\mu})}(\mathrm{U}(\mathcal{R}W'))$, where

$$\theta(\Psi_{M,\mu}) = \chi \cdot \mu^{-n} \cdot \chi_W^{-1}\chi_V + \chi_V \boxtimes S_{n-1} + \mu \cdot \chi_W^{-1}\chi_V \boxtimes S_n.$$

Next, we shall use an analog of the seesaw diagram (♮.1). The following is an analog of the key Proposition 3.2.

Proposition 3.3. *Let V' be the $(n+1)$-dimensional skew-Hermitian space of sign*

$$\epsilon(V') = \epsilon\left(V^\flat\right) \cdot \epsilon(\mathcal{R}W') \cdot \epsilon\left(\frac{1}{2}, \mathrm{As}^+_{L/E}(M_1) \cdot \mu^{-1}, \psi_{E,\delta}\right) \cdot \omega_{E/F}(-1)^n \tag{‡}$$

and

$$\omega' = \omega_{V',\mu}[\chi].$$

Then Λ' is the (big) theta lift of ω' to $\mathrm{U}(\mathcal{R}W')$, that is, $\Lambda' = \Theta(\omega')$. Here, we are using the same splitting characters as described in Equation (♮.1).

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Proof. We first check the special case that $n=2$ and $\chi^\flat=\mu^2$. So

$$\Psi_{M,\mu}=\chi_W+\mu\boxtimes S_2$$

and

$$\theta(\Psi_{M,\mu})=\chi_V+\mu\cdot\chi_W^{-1}\chi_V\boxtimes S_2+(\chi_V^c)^\vee.$$

Let V'' and $\mathcal{R}W''$ be the companion spaces of V' and $\mathcal{R}W'$ respectively. Consider the following map given by the theta correspondence:

$$\theta_4:\operatorname{Irr}\mathrm{U}(V')\sqcup\operatorname{Irr}\mathrm{U}(V'')\longrightarrow\operatorname{Irr}\mathrm{U}(\mathcal{R}W')\sqcup\operatorname{Irr}\mathrm{U}(\mathcal{R}W''),$$

where

$$\pi\mapsto\begin{cases}\theta_{\mathcal{R}W'}(\pi)\in\operatorname{Irr}(\mathrm{U}(\mathcal{R}W'))&\text{if }\theta_{\mathcal{R}W'}(\pi)\neq0;\\ \theta_{\mathcal{R}W''}(\pi)\in\operatorname{Irr}(\mathrm{U}(\mathcal{R}W''))&\text{otherwise.}\end{cases}$$

This map is well defined by the theta dichotomy. Using Theorem 2.1, One can easily check by hand that this map restricts to a bijection

$$\theta_4:\Pi_{\Psi_{M,\mu}}(\mathrm{U}(V'))\sqcup\Pi_{\Psi_{M,\mu}}(\mathrm{U}(V''))\longrightarrow\Pi_{\theta(\Psi_{M,\mu})}(\mathrm{U}(\mathcal{R}W'))\sqcup\Pi_{\theta(\Psi_{M,\mu})}(\mathrm{U}(\mathcal{R}W'')).$$

Hence, Λ' is the (big) theta lift of some

$$\omega'\in\Pi_{\Psi_{M,\mu}}(\mathrm{U}(V'))\sqcup\Pi_{\Psi_{M,\mu}}(\mathrm{U}(V'')).$$

To check that $\omega'=\omega_{V',\mu}[\chi]$, one can use Theorem 2.3 and Lemma 2.6 to compute the character $\eta'\in\operatorname{Irr}A_{\Psi_{M,\mu}}$ associated to ω'. We omit the details.

Excluding the special case above, the theta correspondence between $\mathrm{U}(V')\times\mathrm{U}(\mathcal{R}W')$ is in the situation of Theorem 2.1(2). It follows that the theta lift $\Theta(\omega_{V',\mu}[\chi])$ to $\mathrm{U}(\mathcal{R}W')$ is nonvanishing. So the proof of the general case comes down to a computation of the labellings similar to the proof of Proposition 3.2. We shall not repeat the tedious computation here. $\square$

Now, we can consider the following seesaw diagram, with respect to the same splitting characters as described in Equation (♮.1):

$$\begin{tikzcd}
\mathrm{U}(\mathcal{R}W') \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V'_K) \arrow[d, dash] \arrow[dl, dash] & & \Lambda'=\Theta(\omega_{V',\mu}[\chi]) \arrow[d, dash] \arrow[dr, dash] & \Pi'=\Theta(\Sigma') \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(W') & \mathrm{U}(V') & & \Sigma' & \omega_{V',\mu}[\chi]
\end{tikzcd}$$

Again by the seesaw identity, we have

$$m_{V'}(\Pi',\mu)=\dim\operatorname{Hom}_{\mathrm{U}(W)}(\Lambda',\Sigma')=m_{V^\flat}(\Pi^\flat,\mu)\tag{✠.4}$$

is nonzero. In particular, Π' is nonzero. By [GI16, Sect. 4.6(P2)], we know that Π' is an irreducible tempered representation of $\mathrm{U}(V'_K)$ lies in the L-packet Π_M. The construction above shows the existence of $\Pi'\in\Pi_M$ such that $m_{V'}(\Pi',\mu)\neq0$.

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3.3. Conclusion

In summary, we have shown that:

Proposition 3.4. *Let V_0 be an n-dimensional Hermitian space over E and M_0' a tempered L-parameter for the unitary group $\mathrm{U}(V_{0,K})$. Assume that Conjecture 1.1 holds for the L-parameter M_0'. Then it also holds for the L-parameter of the form*

$$M = M_0' \cdot \mathcal{X} + M_1,$$

where $\mathcal{X}$ is any conjugate symplectic character of $L^\times$, and M_1 is any conjugate self-dual character of $L^\times$ of parity $(-1)^n$.

Proof. As we have explicated above, given such an L-parameter M, one can construct an L-parameter $M^\flat$ of $\mathrm{U}(V_{0,K})$. As long as Conjecture 1.1 holds for the L-parameter $M^\flat$, it also holds for M. On the other hand, from the construction of $M^\flat$, one can see that

$$M^\flat = M_0' \cdot \mathcal{Y}$$

for some conjugate orthogonal character $\mathcal{Y}$ of $L^\times$. Thus, by Lemma 3.1, Conjecture 1.1 holds for $M^\flat$. □

Now, we can prove the main result of this paper.

Proof of Theorem 1.3. Simply note that if M is a summation of conjugate self-dual characters as described in Theorem 1.3, then so is $M^\flat$. □

The reader may notice the similarity of our setup with the paper [Xue23] of Hang Xue, in which he showed the Bessel case of the local GGP conjecture for unitary groups over $\mathbb{R}$. There, he worked also with L-parameters M of the same form as those in Theorem 1.3. Indeed, we are partly inspired by his results to consider these M's. However, the inductive argument in our proof is different from that in [Xue23] (not to mention that the setting of our result is different).

We end up this paper with a remark on the global conjecture [GGP23, Conj. 11.1]. One can expect to prove the global conjecture for the near equivalence class

$$M = M_1 + \cdots + M_n$$

with each M_i conjugate self-dual automorphic character of GL_1 of parity $(-1)^{n-1}$ by using the same argument. Instead of the Adams' conjecture used in this paper, one will need to show an analog of the Siegel–Weil formula in the global case so that one can compare the theta integrals of $\omega_{V,\mu}$ and $\omega_{V^\flat,\mu}$. More precisely, let Ω_V and $\Omega_{V^\flat}$ be the Weil representation associated to $\mathrm{U}(V) \times \mathrm{U}(\mathcal{R}W)$ and $\mathrm{U}(V^\flat) \times \mathrm{U}(\mathcal{R}W)$, respectively, one needs to compare

$$\int_{[\mathrm{U}(V)]} \theta_\varphi(g,h) f(g)\, dg \quad \text{for } \varphi \in \Omega_V,\ f \in \omega_{V,\mu},\ g \in \mathrm{U}(V),\ h \in \mathrm{U}(\mathcal{R}W)$$

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and

$$\int_{[\mathrm{U}(V^\flat)]} \theta_{\varphi'}(g',h)f'(g')\,dg' \quad \textit{for } \varphi' \in \Omega_{V^\flat},\ f' \in \omega_{V^\flat,\mu},\ g' \in \mathrm{U}(V^\flat),\ h \in \mathrm{U}(\mathcal{R}W).$$

Unfortunately, these theta integrals diverge in general. So one has to properly regularize these theta integrals first. Once a global analog of Proposition 3.2 has been established, the remaining parts should go over smoothly.

Acknowledgements. Both authors were supported by a Singapore government MOE Tier One grant R-146-000-320-114 during the course of this work. The work was completed when the second author visited the Erwin Schrodinger Institute (ESI) in Vienna in April 2022; he thanks the ESI for excellent working conditions and inspiring atmosphere. The authors thank Petar Bakić, Marcela Hanzer and Jialiang Zou for helpful discussion and comments.

Competing interests. The authors have no competing interest to declare.

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