INTEGRATION OF ORBITAL MOTIONS WITH CHEBYSHEV POLYNOMIALS

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Abstract. In this communication, we use Chebyshev series for integrating orbital motions. The nice properties that Chebyshev polynomials have, such as giving good approximations of functions in the Chebyshev norm, easy handling of their algebra with algebraic manipulators, allowing very big step sizes for integration and giving the solution in the form of polynomials, make these polynomials very attractive in orbit computations.

1. Introduction

Let us consider the following initial value problem (IVP)

$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{f}(\boldsymbol{y}(t); t); \qquad \boldsymbol{y}(T_0) = \boldsymbol{y}_0, \tag{1}$$

where $\boldsymbol{y} \in \mathbf{R}^s$ and $\boldsymbol{f} : \mathbf{R}^s \times [T_0, T_f] \longrightarrow \mathbf{R}^s$ is a function of class C^1. We plan to solve this problem by means of a collocation scheme; that is, we will approximate the function $\boldsymbol{f}$ by an adequate interpolation polynomial and afterwards, integrating this polynomial, we will have an approximation of the solution (Clenshaw and Norton, 1963). One of the main advantages of this scheme consists in having the solution and its derivative in the form of a polynomial (that is, a dense output) at very low computational cost.

The next decision to be made is what type of polynomial basis to use. Our choice is the Chebyshev polynomials for several reasons; for instance, their orthogonality, their superior convergence rate properties and their near minimax error nature in the interval of interpolation.

Carpenter (1966) was a pioneer in integrating planetary equations by Chebyshev series. Following Clenshaw's method, Carpenter develops a least

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squares approximation of the right hand members of the planetary equations, and then, integrates formally the resulting Chebyshev series in an iterative way. More recently, Chebyshev series have been used in artificial satellite theory, see e.g. (Belikov, 1993; Agnese *et al.*, 1995; Trubitsina, 1996; Barrio, 1996).

In the work here presented we use an implicit method; the one that Broucke (Broucke, 1969) named after Dziobek–Brouwer. Essentially, it consists in a modification of the Encke formalism. With this method, we perform several integrations for different artificial satellites, taking as integration step size the orbital period.

2. Why Chebyshev Polynomials ?

One of the reasons for using Chebyshev polynomials is that they provide a 'near best' uniform approximation of a function. We say that a polynomial $p_n^* \in \mathcal{P}_n$ is *the best approximation* of a function $f \in \mathcal{C}[-1,1]$ in the l_∞ norm, if $\| f - p_n^* \|_\infty \leq \| f - p_n \|_\infty, \quad \forall p_n \in \mathcal{P}_n$.

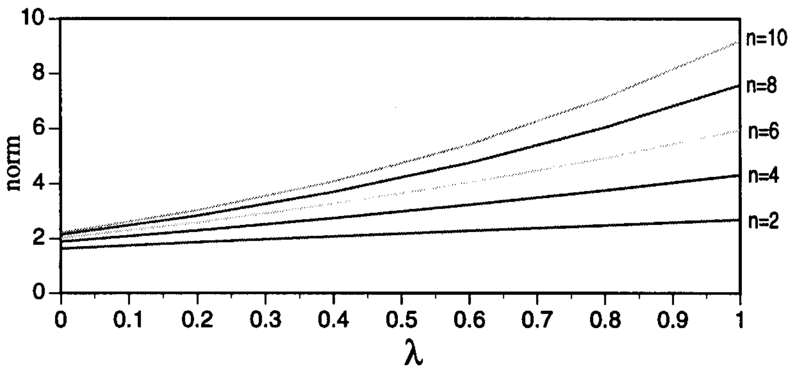

Figure 1. $\| L_n^\lambda \|_\infty$ for several ultraspherical polynomials C_n^λ and for different degrees n.

Let us suppose we have an approximation of a function; the following theorem gives us an estimation of the *goodness* of this approximation with respect to *the best approximation* above defined.

Theorem: *Let $p_n^* \in \mathcal{P}_n[-1,1]$ be the best approximation to $f \in \mathcal{C}[-1,1]$ in the norm l_∞. Then, for all projections $L_n : \mathcal{C}[-1,1] \longrightarrow \mathcal{P}_n[-1,1]$ there holds that*

$$\| f - L_n f \|_\infty \leq (1 + \| L_n \|_\infty) \; \| f - p_n^* \|_\infty .$$

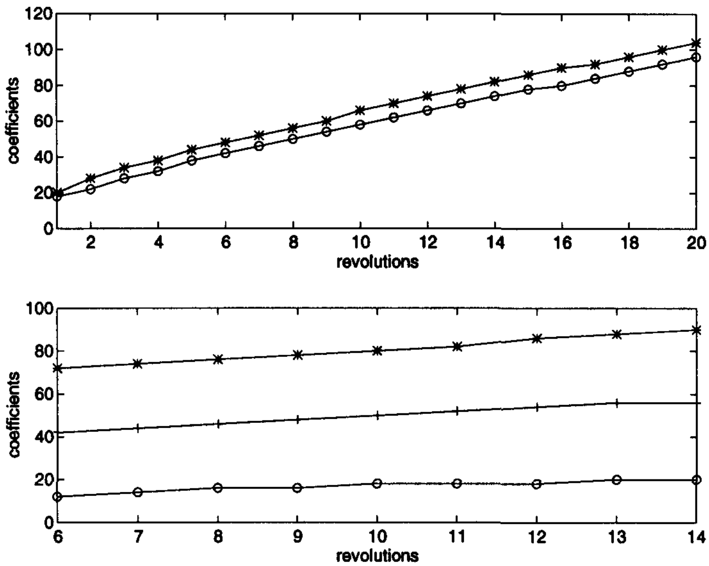

Figure 2. Graphic at the top: order of the polynomials to reach precision of 10(o) and 14(*) digits. Graphic at the bottom: order of the polynomials necessary to approximate 1(o), 8(+) and 16(*) periods.

In Figure 1 we plotted $\| L_n^\lambda \|_\infty$ for several expansions in ultraspherical polynomials C_n^λ and for different degrees n. We see that for $\lambda = 0$, that is, for Chebyshev projection we obtain the minimum values of the norm.

Once we decided to use Chebyshev polynomials, how do we fix the degree of the polynomials to use? We find a hint in Carpenter's work (1966). Since our perturbation forces are periodic or mixed functions, we need to know the behavior of the trigonometric cosine and sine. For instance, for the cosine function (analogously for the sine), its Chebyshev series expansion is

$$\cos(px) = \sum_{r=0}^{\infty}{}' C_{2r}(p) T_{2r}(x), \quad \text{with} \quad C_i(p) = (-1)^{i/2} 2 J_i(p),$$

where $J_i(p)$ is the Bessel function of the first kind. Thus, for a fixed argument p, we need to know how big is i to reach a certain precision required. By considering a time interval Δt (the step size) and a term $\cos 2\pi t/T$ with period T, this term will be a linear combination of trigonometric functions whose argument is px, with $p = \pi \Delta t/T$. We made several numerical examples that appear on Figure 2.

3. Integration Method and Equations of Motion

The method we use is based on the work outlined by Clenshaw (Clenshaw and Norton, 1963). It consists of approximating the right-hand member of the differential system (1) $\boldsymbol{f}(\boldsymbol{y}(t);t)$ by a truncated series of Chebyshev polynomials of the first kind at each step of integration $[t_i, t_{i+1}]$:

$$\boldsymbol{f}(\boldsymbol{y}(t);t) \approx \sum_{n=0}^{m} \boldsymbol{C}_n T_n(u), \qquad -1 \leq u \leq 1, \tag{2}$$

where the change of variable $u = [2t - (t_{i+1} + t_i)]/(t_{i+1} - t_i)$ is needed to transform the interval $[t_i, t_{i+1}]$ into the normal interval $[-1, 1]$. The coefficients $\boldsymbol{C}_n$ are computed by the Chebyshev–Gauss–Lobatto formula [see e.g. Fox and Parker (1968)]

$$\boldsymbol{C}_r \approx \frac{\zeta_k}{m} {\sum_{k=0}^{m}}'' \boldsymbol{f}(\cos(k\pi/m)) \cos\frac{rk\pi}{m}, \quad \text{with} \quad \zeta_k = \begin{cases} 1, & k = 0, m, \\ 2, & 0 < k < m, \end{cases} \tag{3}$$

where the double prime on the summation symbol indicates that both, the coefficients of order zero and order m have to be halved.

Let us recall that since we take only m terms in both sums, in the truncated series and in the evaluation of the coefficients, the polynomial (2) is just the Lagrange interpolation polynomial at the considered points.

Once the approximation polynomial is obtained, we integrate it, and the result is another Chebyshev series

$$\boldsymbol{y}_0 + \frac{t_{i+1} - t_i}{2} \int_{-1}^{u} \sum_{n=0}^{m} \boldsymbol{C}_n T_n(x) \mathrm{d}x = \sum_{n=0}^{m+1} \boldsymbol{b}_n T_n(u), \tag{4}$$

which coefficients are

$$\boldsymbol{b}_{m+1} = \frac{t_{i+1} - t_i}{2} \frac{\boldsymbol{C}_m}{2(m+1)}, \qquad \boldsymbol{b}_m = \frac{t_{i+1} - t_i}{2} \frac{\boldsymbol{C}_{m-1}}{2m},$$

$$\boldsymbol{b}_r = \frac{t_{i+1} - t_i}{2} \frac{1}{2r} (\boldsymbol{C}_{r-1} - \boldsymbol{C}_{r+1}), \quad 1 \leq r < m,$$

$$\boldsymbol{b}_0 = \boldsymbol{b}_1 - \boldsymbol{b}_2 + \boldsymbol{b}_3 - \ldots\ldots + (-1)^m \boldsymbol{b}_{m+1} + \boldsymbol{y}_0.$$

However, the value of $\boldsymbol{y}(t)$ at the chosen points is not known and we must iterate the process, that is to say, for a given initial value of $\boldsymbol{y}(t)$ we compute an approximation of $\boldsymbol{f}(\boldsymbol{y}, t)$; we integrate and we obtain a new approximation of $\boldsymbol{y}(t)$, and so on. Several improvements may be applied to this iterative method (such as Newton methods for instance), although when the function $f(y,t)$ is very intricate, the iterations involve a big amount of operations for computing the Jacobian. The iterative process is repeated until the desired convergence is reached.

This iterative procedure requires the solving of a set of implicit equations; therefore, it would be good to formulate the problem in such a way that it is easy to derive a 'good' first step in the iterations.

Let us consider a perturbed Keplerian problem

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\mu\frac{\boldsymbol{r}}{r^3} + \boldsymbol{P}. \tag{5}$$

Let us define M and $\boldsymbol{s}$ as $M = \mu\,\mathrm{Jacobian}\left(\boldsymbol{r}_k/r_k^3\right)$ and $\boldsymbol{s} = \boldsymbol{r} - \boldsymbol{r}_k$, where $\boldsymbol{r}_k$ stands for a Keplerian orbit of reference. Adding the quantity $M\boldsymbol{s}$ to both sides of (5), we have

$$\frac{d^2\boldsymbol{s}}{dt^2} + \mu\left(\frac{\boldsymbol{s}}{r_k^3} - 3\frac{(\boldsymbol{r}_k\cdot\boldsymbol{s})}{r_k^5}\boldsymbol{r}_k\right) = -\mu\left(\frac{\boldsymbol{r}}{r^3} - \frac{\boldsymbol{r}_k}{r_k^3}\right) + M\,\boldsymbol{s} + \boldsymbol{P} = \boldsymbol{Q}(t). \tag{6}$$

Using the classical method of the variation of parameters, the general solution of (6) can be written in the form

$$\boldsymbol{s} = \sum_{i=1}^{6} K_i(t)\frac{\partial\boldsymbol{r}_k}{\partial a_i}, \tag{7}$$

where $\{a_i \,|\, 1 \le i \le 6\}$ is some set of six independent parameters which defines the orbit. Recall that $\partial\boldsymbol{r}_k/\partial a_i$ $(1 \le i \le 6)$ are six independent solutions of the homogeneous part of (6). Now the difficulty consists in computing the coefficients K_i; it is overcome by solving the differential system $\dot{K}_i = \boldsymbol{\nabla}_{\dot{\boldsymbol{r}}_k} a_i \cdot \boldsymbol{Q}$ (in a similar way to the classical Gauss equations). Consequently, the solution is obtained through

$$\boldsymbol{s} = \sum_{i=1}^{6}\frac{\partial\boldsymbol{r}_k}{\partial a_i}\left(\int_{T_0}^{t}(\boldsymbol{\nabla}_{\dot{\boldsymbol{r}}_k} a_i\cdot\boldsymbol{Q})dx\right). \tag{8}$$

The functions $\partial\boldsymbol{r}_k/\partial a_i$, $\boldsymbol{\nabla}_{\dot{\boldsymbol{r}}_k} a_i$ are evaluated at the reference orbit $\boldsymbol{r}_k$; thus, we may expect a small variation in each variable. With this model, the initial values for the first iteration are those of the reference orbit.

As an illustration, we show here the application of this method to an artificial satellite (24×24 harmonics from the GEM9–10 model). The initial conditions of the simulated orbit are $a = 42000\,\mathrm{km}$, $e = 0.0001$, $i = 5°$. We use the orbital period as integration step and take 30 terms in the series. This solution has been compared with an orbit generated by a RK

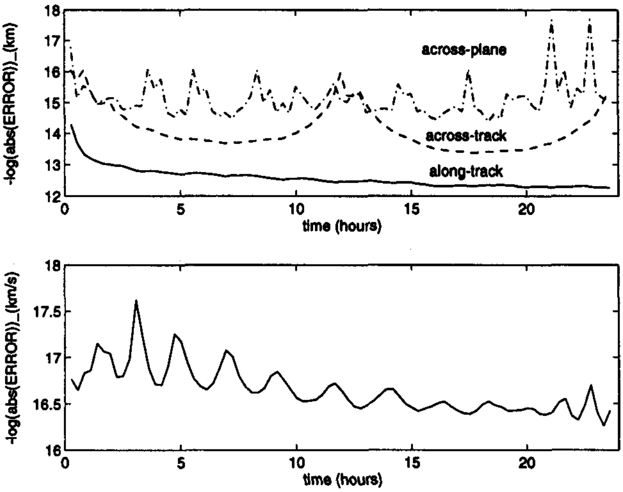

Figure 3. Error in position (along–track, across–track and across–plane components) and velocity (norm).

(DOPRI8) of order eight with an integration step of 100 seconds. The comparison between the two integrations appears in Figure 3. The number of function evaluations are 11232 for the RK and 270 for the method described here.

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