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On algebraic dependencies between Poincaré functions

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Abstract. Let A be a rational function of one complex variable of degree at least two, and z_0 its repelling fixed point with the multiplier λ. A Poincaré function associated with z_0 is a function $\mathcal{P}_{A,z_0,\lambda}$ meromorphic on $\mathbb{C}$ such that $\mathcal{P}_{A,z_0,\lambda}(0) = z_0$, $\mathcal{P}'_{A,z_0,\lambda}(0) \neq 0$, and $\mathcal{P}_{A,z_0,\lambda}(\lambda z) = A \circ \mathcal{P}_{A,z_0,\lambda}(z)$. In this paper, we study the following problem: given Poincaré functions $\mathcal{P}_{A_1,z_1,\lambda_1}$ and $\mathcal{P}_{A_2,z_2,\lambda_2}$, find out if there is an algebraic relation $f(\mathcal{P}_{A_1,z_1,\lambda_1}, \mathcal{P}_{A_2,z_2,\lambda_2}) = 0$ between them and, if such a relation exists, describe the corresponding algebraic curve $f(x, y) = 0$. We provide a solution, which can be viewed as a refinement of the classical theorem of Ritt about commuting rational functions. We also reprove and extend previous results concerning algebraic dependencies between Böttcher functions.

Key words: Poincaré functions, Böttcher functions, linearization
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1. *Introduction*

Let A be a rational function of one complex variable of degree at least two, and z_0 its repelling fixed point with the multiplier λ. We recall that a *Poincaré function* $\mathcal{P}_{A,z_0,\lambda}$ associated with z_0 is a function meromorphic on $\mathbb{C}$ such that $\mathcal{P}_{A,z_0,\lambda}(0) = z_0$, $\mathcal{P}'_{A,z_0,\lambda}(0) \neq 0$, and the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda z} & \mathbb{C} \\ {\scriptstyle \mathcal{P}_{A,z_0,\lambda}} \downarrow & & \downarrow {\scriptstyle \mathcal{P}_{A,z_0,\lambda}} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes. The Poincaré function exists and is defined up to the transformation of argument $z \to cz$, where $c \in \mathbb{C}^*$ (see e.g. [12]). In particular, it is defined in a unique way if to assume that $\mathcal{P}'_{A,z_0,\lambda}(0) = 1$. Such Poincaré functions are called normalized. In this paper, we will consider non-normalized Poincaré functions, so the explicit meaning of the notation $\mathcal{P}_{A,z_0,\lambda}$ is as follows: $\mathcal{P}_{A,z_0,\lambda}$ is *some* meromorphic function satisfying the above conditions. We say that a rational function A is *special* if it is either a Lattès map,

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or it is conjugate to $z^{\pm n}$ or $\pm T_n$. Poincaré functions associated with special functions can be described in terms of classical functions. Moreover, by the result of Ritt [27], these functions are the only Poincaré functions that are periodic.

In this paper, we study the following problem. Let A_1, A_2 be non-special rational functions of degree at least two with repelling fixed points z_1, z_2, and $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ corresponding Poincaré functions. Under what conditions does there exist an algebraic curve $f(x, y) = 0$ such that

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}, \mathcal{P}_{A_2,z_2,\lambda_2}) = 0 \tag{1}$$

and, if such a curve exists, how it can be described? The simplest example of relation (1) is just the equality

$$\mathcal{P}_{A_1,z_0,\lambda_1} = \mathcal{P}_{A_2,z_0,\lambda_2}, \tag{2}$$

which is known to have strong dynamical consequences. Specifically, equality (2) implies easily that A_1 and A_2 commute. On the other hand, by the theorem of Ritt (see [28] and also [6, 23]), every two non-special commuting rational functions of degree at least two have a common iterate. Thus, equality (2) implies that

$$A_1^{\circ l_1} = A_2^{\circ l_2} \tag{3}$$

for some integers $l_1, l_2 \geq 1$. Moreover, the Ritt theorem essentially is equivalent to the statement that equality (2) implies equality (3), since it was observed already by Fatou and Julia [8, 10] that if two rational functions commute, then some of their iterates share a repelling fixed point and a corresponding Poincaré function.

To the best of our knowledge, the problem of describing algebraic dependencies between Poincaré functions has never been considered in the literature. Nevertheless, the problem of describing algebraic dependencies between *Böttcher functions*, similar in spirit, has been investigated previously [2, 14]. We recall that for a polynomial A of degree n, a corresponding Böttcher function $\mathcal{B}_A$ is a Laurent series

$$\mathcal{B}_A = a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \in z\mathbb{C}[[1/z]], \quad a_{-1} \neq 0, \tag{4}$$

that makes the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z^n} & \mathbb{C} \\ \mathcal{B}_A \downarrow & & \downarrow \mathcal{B}_A \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{5}$$

commutative. In this notation, the result of Becker and Bergweiler [2] (see also [3]), states that if A_1 and A_2 are polynomials of the same degree d, then the function $\beta = \mathcal{B}_{A_1} \circ \mathcal{B}_{A_2}^{-1}$ is transcendental, unless either β is linear, or A_1 and A_2 are special (notice that since a polynomial cannot be a Lattès map, a polynomial is special if and only if it is conjugate to z^n or $\pm T_n$). Since the equality

$$f(\mathcal{B}_{A_1}(z), \mathcal{B}_{A_2}(z)) = 0$$

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holds for some $f(x, y) \in \mathbb{C}[x, y]$ if and only if the function β is algebraic, this result implies the absence of algebraic dependencies of degree greater than one between $\mathcal{B}_{A_1}(z)$ and $\mathcal{B}_{A_2}(z)$ for non-special A_1 and A_2 of the same degree.

Subsequently, it was proved by Nguyen in [14] that the equality

$$f(\mathcal{B}_{A_1}(z^{d_1}), \mathcal{B}_{A_2}(z^{d_2})) = 0 \tag{6}$$

holds for some integers $d_1, d_2 \geq 1$ if and only if there exist polynomials X_1, X_2, B and integers $l_1, l_2 \geq 1$ such that the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1^{\circ l_1}, A_2^{\circ l_2})} & (\mathbb{CP}^1)^2 \end{array}$$

commutes. Notice that although the result of Nguyen deals with the more general situation than the result of Becker and Bergweiler, the former does not formally imply the latter.

Let us recall that an algebraic curve $C : f(x, y) = 0$ has genus zero if and only if it admits a parameterization $z \to (X_1(z), X_2(z))$ by rational functions X_1, X_2. Such a parameterization is called *generically one-to-one* if it is one-to-one except for finitely many points. By the Lüroth theorem, this is equivalent to saying that X_1 and X_2 generate the whole field of rational functions $\mathbb{C}(z)$. In this notation, our main result is the following analog of the result of Nguyen.

THEOREM 1.1. *Let A_1, A_2 be non-special rational functions of degree at least two, z_1, z_2 their repelling fixed points with multipliers λ_1, λ_2, and $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ Poincaré functions. Assume that $C : f(x, y) = 0$ is an irreducible algebraic curve, and d_1, d_2 are coprime positive integers such that the equality*

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2})) = 0 \tag{7}$$

holds. Then, C has genus zero. Furthermore, if $C : f(x, y) = 0$ is an irreducible algebraic curve of genus zero with a generically one-to-one parameterization by rational functions $z \to (X_1(z), X_2(z))$, and d_1, d_2 are coprime positive integers, then equality (7) holds for some Poincaré functions $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ if and only if there exist positive integers l_1, l_2, k and a rational function B with a repelling fixed point z_0 such that the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1^{\circ l_1}, A_2^{\circ l_2})} & (\mathbb{CP}^1)^2 \end{array} \tag{8}$$

commutes and the equalities

$$X_1(z_0) = z_1, \quad X_2(z_0) = z_2, \tag{9}$$

$$\operatorname{ord}_{z_0} X_1 = d_1 k, \quad \operatorname{ord}_{z_0} X_2 = d_2 k \tag{10}$$

hold.

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Notice that Theorem 1.1 can be considered as a refinement of the Ritt theorem. Indeed, equality (2) is a particular case of the condition (7), where the curve

$$f(x, y) = x - y = 0$$

is parameterized by the functions $X_1 = z$, $X_2 = z$. Thus, in this case, the diagram (8) reduces to equality (3). More generally, considering the curve $x - R(y) = 0$, where R is a rational function, we conclude that the equality

$$\mathcal{P}_{A_1,z_1,\lambda_1} = R \circ \mathcal{P}_{A_2,z_2,\lambda_2}$$

implies that there exist $l_1, l_2 \geq 1$ such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{A_2^{\circ l_2}} & \mathbb{CP}^1 \\ \downarrow R & & \downarrow R \\ \mathbb{CP}^1 & \xrightarrow{A_1^{\circ l_1}} & \mathbb{CP}^1 \end{array}$$

commutes.

Notice also that Theorem 1.1 implies the following handy criterion for the algebraic independence of Poincaré functions.

COROLLARY 1.2. *Let A_1, A_2 be non-special rational functions of degrees $n_1 \geq 2$, $n_2 \geq 2$, and z_1, z_2 their repelling fixed points with multipliers λ_1, λ_2. Then, Poincaré functions $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ are algebraically independent, unless there exist positive integers l_1, l_2 and l_1', l_2' such that $n_1^{l_1} = n_2^{l_2}$ and $\lambda_1^{l_1'} = \lambda_2^{l_2'}$.*

In addition to Theorem 1.1, we prove the following more precise version of the theorem of Nguyen, which formally includes and generalizes the result of Becker and Bergweiler.

THEOREM 1.3. *Let A_1, A_2 be non-special polynomials of degree at least two, and $\mathcal{B}_{A_1}$, $\mathcal{P}_{A_2}$ Böttcher functions. Assume that $C : f(x, y) = 0$ is an irreducible algebraic curve, and d_1, d_2 are coprime positive integers such that the equality*

$$f(\mathcal{B}_{A_1}(z^{d_1}), \mathcal{B}_{A_2}(z^{d_2})) = 0 \tag{11}$$

holds. Then, C has the form $Y_1(x) - Y_2(y) = 0$, where Y_1, Y_2 are polynomials of coprime degrees, and can be parameterized by polynomials. Furthermore, if $C : f(x, y) = 0$ is an irreducible algebraic curve as above with a generically one-to-one parameterization by polynomials $z \to (X_1(z), X_2(z))$, and d_1, d_2 are coprime positive integers, then equality (11) holds for some Böttcher functions $\mathcal{B}_{A_1}$, $\mathcal{B}_{A_2}$ if and only if there exist positive integers l_1, l_2 and a polynomial B such that the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1^{\circ l_1},A_2^{\circ l_2})} & (\mathbb{CP}^1)^2 \end{array} \tag{12}$$

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commutes, and the equalities

$$\deg X_1 = d_1, \quad \deg X_2 = d_2 \tag{13}$$

hold. In particular, the equality

$$f(\mathcal{B}_{A_1}(z), \mathcal{B}_{A_2}(z)) = 0$$

implies that $C : f(x, y) = 0$ *has degree one and some iterates of* A_1 *and* A_2 *are conjugate.*

Notice that the parameters d_1, d_2 appear in conclusions of both Theorems 1.1 and 1.3. However, the condition (10) is less restrictive than the condition (13). In particular, applying Theorem 1.3 for $d_1 = d_2 = 1$, we conclude that algebraic dependencies between Bötcher functions are essentially trivial. However, algebraic dependencies between Poincaré functions do exist (see §3).

The approach of Nguyen to the study of algebraic dependencies (6) relies on the fact that such dependencies give rise to *invariant algebraic curves* for endomorphisms

$$(A_1, A_2) : (\mathbb{CP}^1)^2 \to (\mathbb{CP}^1)^2, \tag{14}$$

given by the formula

$$(z_1, z_2) \to (A_1(z_1), A_2(z_2)), \tag{15}$$

where A_1 and A_2 are polynomials. Say, for A_1 and A_2 of the same degree n, this can be seen immediately, since after substituting z^n for z into equation (6), we obtain the equality

$$f(A_1 \circ \mathcal{B}_{A_1}(z^{d_1}), A_2 \circ \mathcal{B}_{A_2}(z^{d_2})) = 0,$$

implying that $f(x, y) = 0$ is (A_1, A_2)-invariant. Invariant curves for polynomial endomorphisms of the form (14) were classified by Medvedev and Scanlon [**11**], and the proof of the theorem of Nguyen relies crucially on this classification.

Our approach to the study of algebraic dependencies (1) is similar. However, instead of the paper [**11**], we use the results of the recent paper [**25**] providing a classification of invariant curves for endomorphisms (15) defined by arbitrary non-special *rational* functions A_1, A_2. Notice that [**11**] is based on the Ritt theory of polynomial decompositions [**26**], which does not extend to rational functions. Accordingly, the approach of [**25**] is completely different and relies on the recent results [**16**, **18**–**21**] about *semiconjugate rational functions*, which appear naturally in a variety of different contexts (see e.g. [**4**, **7**, **9**, **11**, **14**, **17**, **20**, **22**, **25**]).

This paper is organized as follows. In §2, we review the notion of a *generalized Lattès map*, introduced in [**20**], and recall some results about semiconjugate rational functions and invariant curves proved in [**25**]. In §3, we prove Theorem 1.1. We also show that for rational functions that are not generalized Lattès maps, equality (7) under the condition $\mathrm{GCD}(d_1, d_2) = 1$ implies the equality $d_1 = d_2 = 1$ (Theorem 3.6). Finally, in §4, based on results of [**17**], which complements some of results of [**11**], we reconsider algebraic dependencies between Böttcher functions and prove Theorem 1.3.

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2. *Generalized Lattès maps and invariant curves*

2.1. *Generalized Lattès maps and semiconjugacies.* Let us recall that *a Riemann surface orbifold* is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface R and a ramification function $\nu : R \to \mathbb{N}$, which takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$, the *Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left(\frac{1}{\nu(z)} - 1 \right).$$

For orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we write $\mathcal{O}_1 \preceq \mathcal{O}_2$ if $R_1 = R_2$ and for any $z \in R_1$, the condition $\nu_1(z) \mid \nu_2(z)$ holds.

Let $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ be orbifolds, and let $f : R_1 \to R_2$ be a holomorphic branched covering map. We say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *covering map between orbifolds* if for any $z \in R_1$, the equality

$$\nu_2(f(z)) = \nu_1(z) \deg_z f$$

holds, where $\deg_z f$ is the local degree of f at the point z. If for any $z \in R_1$ the weaker condition

$$\nu_2(f(z)) \mid \nu_1(z) \deg_z f \tag{16}$$

is satisfied, we say that $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a *holomorphic map between orbifolds*. If $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds with compact supports, then the Riemann–Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2) \deg f. \tag{17}$$

More generally, if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a holomorphic map, then

$$\chi(\mathcal{O}_1) \le \chi(\mathcal{O}_2) \deg f, \tag{18}$$

and the equality is attained if and only if $f : \mathcal{O}_1 \to \mathcal{O}_2$ is a covering map between orbifolds (see [16, Proposition 3.2]).

Let R_1, R_2 be Riemann surfaces and $f : R_1 \to R_2$ a holomorphic branched covering map. Assume that R_2 is provided with a ramification function ν_2. To define a ramification function ν_1 on R_1 so that f would be a holomorphic map between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$, we must satisfy the condition (16), and it is easy to see that for any $z \in R_1$, a minimum possible value for $\nu_1(z)$ is defined by the equality

$$\nu_2(f(z)) = \nu_1(z) \mathrm{GCD}(\deg_z f, \nu_2(f(z))). \tag{19}$$

In the case where equation (19) is satisfied for any $z \in R_1$, we say that f is *a minimal holomorphic map* between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$.

We recall that *a Lattès map* can be defined as a rational function A such that $A : \mathcal{O} \to \mathcal{O}$ is a covering self-map for some orbifold $\mathcal{O}$ on $\mathbb{CP}^1$ (see [13, 20]). Thus, A is a Lattès map if there exists an orbifold $\mathcal{O} = (\mathbb{CP}^1, \nu)$ such that for any $z \in \mathbb{CP}^1$, the equality

$$\nu(A(z)) = \nu(z) \deg_z A$$

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holds. By equality (17), such $\mathcal{O}$ necessarily satisfies $\chi(\mathcal{O}) = 0$. Following [20], we say that a rational function A of degree at least two is a *generalized Lattès map* if there exists an orbifold $\mathcal{O} = (\mathbb{CP}^1, \nu)$, distinct from the non-ramified sphere, such that $A : \mathcal{O} \to \mathcal{O}$ is a minimal holomorphic self-map between orbifolds; that is, for any $z \in \mathbb{CP}^1$, the equality

$$\nu(A(z)) = \nu(z)\mathrm{GCD}(\deg_z A, \nu(A(z)))$$

holds. By inequality (18), such $\mathcal{O}$ satisfies $\chi(\mathcal{O}) \geq 0$. Notice that any special rational function is a generalized Lattès map, and that some iterate $A^{\circ l}$, $l \geq 1$, of a rational function A is a generalized Lattès map if and only if A is a generalized Lattès map (see [25, §2.3].

Generalized Lattès maps are closely related to the problem of describing semiconjugate rational functions, that is, rational functions that make the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ X\downarrow & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{20}$$

commutative. For a general theory, we refer the reader to [16, 18–21]. Below, we need only the following two results, which are simplified reformulations of [25, Proposition 3.3 and Theorem 4.14].

The first result states that if the function A in diagram (20) is not a generalized Lattès map, then diagram (20) can be completed to a diagram of the very special form.

PROPOSITION 2.1. *Let A be a rational function of degree at least two that is not a generalized Lattes map, and X, B rational functions such that the diagram (20) commutes. Then there exists a rational function Y such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ X\downarrow & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \\ Y\downarrow & & \downarrow Y \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \end{array}$$

commutes, and the equalities

$$Y \circ X = B^{\circ d}, \quad X \circ Y = A^{\circ d},$$

hold for some $d \geq 0$.

The second result relates an arbitrary non-special rational function with some rational function that is not a generalized Lattès map through the semiconjugacy relation.

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THEOREM 2.2. *Let A be a non-special rational function of degree at least two. Then there exist rational functions θ and F such that F is not a generalized Lattès map and the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{F} & \mathbb{CP}^1 \\ \theta\downarrow & & \downarrow\theta \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes.

2.2. *Invariant curves.* Let A_1, A_2 be rational functions, (A_1, A_2) the map given by formulas (14) and (15), and C an irreducible algebraic curve in $(\mathbb{CP}^1)^2$. We say that C is (A_1, A_2)- *invariant* if $(A_1, A_2)(C) = C$. We recall that a *desingularization* of C is a compact Riemann surface $\widetilde{C}$ together with a map $\pi : \widetilde{C} \to C$, which is biholomorphic except for finitely many points.

The simplest (A_1, A_2)-invariant curves are vertical lines $x = a$, where a is a fixed point of A_1, and horizontal lines $y = b$, where b is a fixed point of A_2. Other invariant curves are described as follows (see [25, Theorem 4.1]).

THEOREM 2.3. *Let A_1, A_2 be rational functions of degree at least two, and C an irreducible (A_1, A_2)-invariant curve that is not a vertical or horizontal line. Then the desingularization $\widetilde{C}$ of C has genus zero or one, and there exist non-constant holomorphic maps $X_1, X_2 : \widetilde{C} \to \mathbb{CP}^1$ and $B : \widetilde{C} \to \widetilde{C}$ such that the diagram*

$$\begin{array}{ccc} (\widetilde{C})^2 & \xrightarrow{(B,B)} & (\widetilde{C})^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{CP}^1)^2 \end{array}$$

commutes and the map $t \to (X_1(t), X_2(t))$ is a generically one-to-one parameterization of C. Finally, unless both A_1, A_2 are Lattès maps, $\widetilde{C}$ has genus zero.

For a general description of (A_1, A_2)-invariant curves, we refer the reader to [25]. Below, we need only the following description of invariant curves in the case where $A_1 = A_2$ (see [25, Theorem 1.2]).

THEOREM 2.4. *Let A be a rational function of degree at least two that is not a generalized Lattès map, and C an irreducible algebraic curve in $(\mathbb{CP}^1)^2$ that is not a vertical or horizontal line. Then, C is (A, A)-invariant if and only if there exist rational functions U_1, U_2, V_1, V_2 commuting with A such that the equalities*

$$U_1 \circ V_1 = U_2 \circ V_2 = A^{\circ d},$$
$$V_1 \circ U_1 = V_2 \circ U_2 = A^{\circ d}$$

hold for some $d \geq 0$ and the map $t \to (U_1(t), U_2(t))$ is a parameterization of C.

Notice that, in general, the parameterization $t \to (U_1(t), U_2(t))$ provided by Theorem 2.4 is not generically one-to-one.

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3. *Algebraic dependencies between Poincaré functions*

Our proof of Theorem 1.1 is based on the results of §2 and the lemmas below.

LEMMA 3.1. *Let $C: \ f(x,y)=0$ be an irreducible algebraic curve that admits a parameterization $z \to (\varphi_1(z), \varphi_2(z))$ by functions meromorphic on $\mathbb{C}$. Then the desingularization $\widetilde{C}$ of C has genus zero or one, and there exist meromorphic functions $\varphi: \mathbb{C} \to \widetilde{C}$ and $\widetilde{\varphi}_1: \widetilde{C} \to \mathbb{CP}^1$, $\widetilde{\varphi}_2: \widetilde{C} \to \mathbb{CP}^1$ such that*

$$\varphi_1 = \widetilde{\varphi}_1 \circ \varphi, \quad \varphi_2 = \widetilde{\varphi}_2 \circ \varphi,$$

and the map $z \to (\widetilde{\varphi}_1(z), \widetilde{\varphi}_2(z))$ from $\widetilde{C}$ to C is generically one-to-one.

Proof. The lemma follows from the Picard theorem (see [**1**, Theorems 1 and 2]). □

LEMMA 3.2. *Let A be a non-special rational function of degree at least two, and z_0 its fixed point with the multiplier λ. Assume that W is a rational function of degree at least two commuting with A such that z_0 is a fixed point of W with the multiplier μ. Then there exist positive integers l and k such that $\mu^l = \lambda^k$.*

Proof. By the theorem of Ritt, there exist positive integers l and k such that $W^{\circ l} = A^{\circ k}$, and differentiating this equality at z_0, we conclude that $\mu^l = \lambda^k$. □

LEMMA 3.3. *Let A, B be rational functions of degree at least two, and X a non-constant rational function such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

commutes. Assume that z_0 is a fixed point of B with the multiplier λ_0. Then $z_1 = X(z_0)$ is a fixed point z_1 of A with the multiplier

$$\lambda_1 = \lambda_0^{\operatorname{ord}_{z_0} X}. \tag{21}$$

In particular, z_0 is a repelling fixed point of B if and only if z_1 is a repelling fixed point of A. Furthermore, if z_0 is repelling and $\mathcal{P}_{B,z_0,\lambda}$ is a Poincaré function, then the equality

$$\mathcal{P}_{A,z_1,\lambda_1}(z^{\operatorname{ord}_{z_0} X}) = X \circ \mathcal{P}_{B,z_0,\lambda_0} \tag{22}$$

holds for some Poincaré function $\mathcal{P}_{A,z_1,\lambda_1}$.

Proof. It is clear that z_1 is a fixed point of A, and a local calculation shows that equality (21) holds. Thus, z_1 is a repelling fixed point of A if and only if z_0 is a repelling fixed point of B.

The rest of the proof is obtained by a modification of the proof of the uniqueness of a Poincaré function (see e.g. [**12**]). Namely, considering the function

$$G = \mathcal{P}^{-1}_{A,z_1,\lambda_1} \circ X \circ \mathcal{P}_{B,z_0,\lambda_0}$$

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holomorphic in a neighborhood of zero and satisfying $G(0) = 0$, we see that

$$\begin{aligned} G(\lambda_0 z) = \mathcal{P}^{-1}_{A,z_1,\lambda_1} \circ X \circ B \circ \mathcal{P}_{B,z_0,\lambda_0} &= \mathcal{P}^{-1}_{A,z_1,\lambda_1} \circ A \circ X \circ \mathcal{P}_{B,z_0,\lambda_0} \\ &= \lambda_1 \circ \mathcal{P}^{-1}_{A,z_1,\lambda_1} \circ X \circ \mathcal{P}_{B,z_0,\lambda_0} = \lambda_0^{\operatorname{ord}_{z_0} X} G(z). \end{aligned}$$

Comparing now coefficients of the Taylor expansions in the left and the right parts of this equality, and taking into account that λ_0 is not a root of unity, we conclude that $G = z^{\operatorname{ord}_{z_0} X}$ for some choice of $\mathcal{P}_{A,z_1,\lambda_1}$, implying equality (22). □

LEMMA 3.4. *Let A be a rational function of degree at least two, z_0 its repelling fixed point with the multiplier λ, and $\mathcal{P}_{A,z_0,\lambda}$ a Poincaré function. Assume that $C : f(x, y) = 0$ is an irreducible algebraic curve, and d_1, d_2 are positive integers such that the equality*

$$f(\mathcal{P}_{A,z_0,\lambda_0}(z^{d_1}), \mathcal{P}_{A,z_0,\lambda_0}(z^{d_2})) = 0 \tag{23}$$

holds. Then $d_1 = d_2$, and C is the diagonal $x = y$.

Proof. Since

$$z \to (\mathcal{P}_{A,z_0,\lambda_0}(z^{d_1}), \mathcal{P}_{A,z_0,\lambda_0}(z^{d_2})) \tag{24}$$

is a parameterization of C, it is clear that C is not a vertical or horizontal line. Furthermore, substituting $\lambda_0 z$ for z into equality (23), we see that the curve C is $(A^{\circ d_1}, A^{\circ d_2})$-invariant. Therefore, by Theorem 2.3, there exist non-constant holomorphic maps $X_1, X_2 : \widetilde{\mathcal{C}} \to \mathbb{CP}^1$ and $B : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} (\widetilde{\mathcal{C}})^2 & \xrightarrow{(B,B)} & (\widetilde{\mathcal{C}})^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A^{\circ d_1},A^{\circ d_2})} & (\mathbb{CP}^1)^2 \end{array}$$

commutes. Thus,

$$\deg A^{\circ d_1} = \deg A^{\circ d_2} = \deg B,$$

and hence $d_1 = d_2$. Since the parameterization of C has the form (24), this implies that C is the diagonal. □

COROLLARY 3.5. *Let A_1, A_2 be rational functions of degree at least two, z_1, z_2 their repelling fixed points with multipliers λ_1, λ_2, and $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ Poincaré functions. Assume that $C : f(x, y) = 0$ is an irreducible algebraic curve and $d_1, d_2, \widetilde{d}_1, \widetilde{d}_2$ are positive integers such that* $\mathrm{GCD}(d_1, d_2) = 1$ *and the equalities*

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2})) = 0, \tag{25}$$

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{\widetilde{d}_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{\widetilde{d}_2})) = 0 \tag{26}$$

hold. Then there exists a positive integer k such that the equalities

$$\widetilde{d}_1 = kd_1, \quad \widetilde{d}_2 = kd_2 \tag{27}$$

hold.

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Proof. It is clear that equalities (25) and (26) imply the equalities

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1\widetilde{d}_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2\widetilde{d}_1})) = 0$$

and

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1\widetilde{d}_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_1\widetilde{d}_2})) = 0.$$

Eliminating now from these equalities $\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1\widetilde{d}_1})$, we conclude that the functions $\mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2\widetilde{d}_1})$ and $\mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_1\widetilde{d}_2})$ are algebraically dependent. Therefore, $\widetilde{d}_1 d_2 = d_1\widetilde{d}_2$ by Lemma 3.4, implying equalities (27). □

Proof of Theorem 1.1. Let $C : f(x, y) = 0$ be an irreducible algebraic curve with a generically one-to-one parameterization by rational functions $z \to (X_1(z), X_2(z))$, and d_1, d_2 coprime positive integers. Assume that the diagram (8) commutes for some rational function B with a repelling fixed point z_0, and equalities (9) and (10) hold. Then, denoting the multiplier of z_0 by λ and using Lemma 3.3, we see that

$$\lambda_1^{l_1} = \lambda^{\mathrm{ord}_{z_0} X_1}, \quad \lambda_2^{l_2} = \lambda^{\mathrm{ord}_{z_0} X_2}, \tag{28}$$

and

$$\begin{aligned} 0 &= f(X_1, X_2) = f(X_1 \circ \mathcal{P}_{B,z,\lambda}, X_2 \circ \mathcal{P}_{B,z,\lambda}) \\ &= f(\mathcal{P}_{A_1^{\circ l_1},z_1,\lambda_1^{l_1}}(z^{\mathrm{ord}_{z_0} X_1}), \mathcal{P}_{A_2^{\circ l_2},z_2,\lambda_2^{l_2}}(z^{\mathrm{ord}_{z_0} X_2})). \end{aligned}$$

Since

$$\mathcal{P}_{A_1^{\circ l_1},z_1,\lambda_1^{l_1}}(z) = \mathcal{P}_{A_1,z_1,\lambda_1}(z), \quad \mathcal{P}_{A_2^{\circ l_2},z_2,\lambda_2^{l_2}}(z) = \mathcal{P}_{A_2,z_2,\lambda_2}(z),$$

this implies that

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{\mathrm{ord}_{z_0} X_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{\mathrm{ord}_{z_0} X_2})) = 0. \tag{29}$$

Finally, equality (10) implies that if equality (29) holds, then equality (7) also holds. This proves the 'if' part of the theorem.

To prove the 'only if' part, it is enough to show that equality (7) implies that there exist positive integers r_1, r_2 such that

$$\lambda_1^{r_1} = \lambda_2^{r_2} = \lambda. \tag{30}$$

Indeed, in this case, substituting λz for z into equality (7), we obtain the equality

$$f(A_1^{\circ d_1 r_1} \circ \mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1}), A_2^{\circ d_2 r_2} \circ \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2})) = 0.$$

Therefore, for

$$l_1 = d_1 r_1, \quad l_2 = d_2 r_2,$$

the curve C is $(A_1^{\circ l_1}, A_2^{\circ l_2})$-invariant, implying by Theorem 2.3 that C has genus zero and there exist rational functions X_1, X_2 and B such that the diagram (8) commutes and the map $z \to (X_1(z), X_2(z))$ is a generically one-to-one parameterization of C.

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It follows now from Lemma 3.1 that there exists a meromorphic function φ such that the equalities

$$\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1}) = X_1 \circ \varphi(z), \quad \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2}) = X_2 \circ \varphi(z)$$

hold. Thus,

$$z_1 = \mathcal{P}_{A_1,z_1,\lambda_1}(0) = X_1 \circ \varphi(0), \quad z_2 = \mathcal{P}_{A_2,z_2,\lambda_2}(0) = X_2 \circ \varphi(0),$$

implying that equalities (9) hold for the point $z_0 = \varphi(0)$.

Further, since z_1 and z_2 are fixed points of A_1 and A_2, the point z_0 is a preperiodic point of B. Thus, changing in equation (8) the functions B and $A_1^{\circ l_1}$, $A_2^{\circ l_2}$ to some of their iterates, and the point z_0 to some point in its B-orbit, we may assume that z_0 is a fixed point of B. Moreover, z_0 is repelling by Lemma 3.3. Let us recall now that, by what is proved above, equalities (8) and (9) imply equality (29). Thus, equalities (7) and (29) hold simultaneously and hence equality (10) holds by Corollary 3.5.

Let us show now that equality (7) implies equality (30). Assume first that A_1 and A_2 are not generalized Lattès maps. Substituting $\lambda_2 z$ for z into equality (7), we obtain the equality

$$\begin{aligned} &f(\mathcal{P}_{A_1,z_1,\lambda_1} \circ (\lambda_2 z)^{d_1}, \mathcal{P}_{A_2,z_2,\lambda_2} \circ (\lambda_2 z)^{d_2}) \\ &\quad = f(\mathcal{P}_{A_1,z_1,\lambda_1} \circ (\lambda_2 z)^{d_1}, A_2^{\circ d_2} \circ \mathcal{P}_{A_2,z_2,\lambda_2} \circ z^{d_2}) = 0, \end{aligned}$$

implying that the functions $\mathcal{P}_{A_1,z_1,\lambda_1} \circ (\lambda_2 z)^{d_1}$ and $\mathcal{P}_{A_2,z_2,\lambda_2} \circ z^{d_2}$ satisfy the equality

$$g(\mathcal{P}_{A_1,z_1,\lambda_1} \circ (\lambda_2 z)^{d_1}, \mathcal{P}_{A_2,z_2,\lambda_2} \circ z^{d_2}) = 0, \tag{31}$$

where $g(x, y) = f(x, A_2^{\circ d_2}(y))$. Eliminating now from equalities (7) and (31) the function $\mathcal{P}_{A_2,z_2,\lambda_2} \circ z^{d_2}$, we conclude that the functions $\mathcal{P}_{A_1,z_1,\lambda_1} \circ z^{d_1}$ and $\mathcal{P}_{A_1,z_1,\lambda_1} \circ (\lambda_2 z)^{d_1}$ are algebraically dependent. In turn, this implies that the functions $\mathcal{P}_{A_1,z_1,\lambda_1}(z)$ and $\mathcal{P}_{A_1,z_1,\lambda_1}(\lambda_2^{d_1} z)$ also are algebraically dependent.

Let $\widetilde{C} : \widetilde{f}(x, y) = 0$ be a curve such that

$$\widetilde{f}(\mathcal{P}_{A_1,z_1,\lambda_1}(z), \mathcal{P}_{A_1,z_1,\lambda_1}(\lambda_2^{d_1} z)) = 0.$$

Then, substituting $\lambda_1 z$ for z, we see that $\widetilde{f}$ is (A_1, A_1)-invariant. Therefore, by Theorem 2.4, there exist rational functions V_1 and V_2 commuting with A_1 such that $\widetilde{C}$ is a component of the curve

$$V_1(x) - V_2(y) = 0,$$

implying that the equality

$$V_1 \circ \mathcal{P}_{A_1,z_1,\lambda_1}(z) = V_2 \circ \mathcal{P}_{A_1,z_1,\lambda_1}(\lambda_2^{d_1} z) \tag{32}$$

holds. Furthermore, it follows from the Ritt theorem that there exist positive integers s_1, s_2, and s such that

$$V_1^{\circ s_1} = V_2^{\circ s_2} = A_1^{\circ s}. \tag{33}$$

Since equality (32) implies that for every $l \geq 1$, the equality

$$V_1^{\circ l} \circ V_1 \circ \mathcal{P}_{A_1,z_1,\lambda_1}(z) = V_1^{\circ l} \circ V_2 \circ \mathcal{P}_{A_1,z_1,\lambda_1}(\lambda_2^{d_1} z)$$

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holds, setting

$$W_1 = V_1^{\circ s_1}, \quad W_2 = V_1^{\circ(s_1-1)} \circ V_2,$$

we see that W_1 and W_2 also commute with A_1 and satisfy

$$W_1 \circ \mathcal{P}_{A_1,z_1,\lambda_1}(z) = W_2 \circ \mathcal{P}_{A_1,z_1,\lambda_1}(\lambda_2^{d_1} z). \tag{34}$$

In addition, z_1 is a fixed point of W_1 by equality (33). Finally, since equality (34) implies the equality

$$W_1(z_1) = W_2(z_1),$$

the point z_1 is also a fixed point of W_2.

Differentiating equality (34) at zero, we see that the multipliers

$$\mu_1 = W_1'(z_1), \quad \mu_2 = W_2'(z_1)$$

satisfy the equality

$$\mu_1 = \mu_2 \lambda_2^{d_1}. \tag{35}$$

On the other hand, Lemma 3.2 yields that there exist positive integers k_1, k_2, and k such that

$$\mu_1^{k_1} = \mu_2^{k_2} = \lambda_1^k. \tag{36}$$

It follows now from equalities (35) and (36) that

$$\lambda_1^{kk_2} = \mu_1^{k_1k_2} = \mu_2^{k_1k_2}\lambda_2^{d_1k_1k_2} = \lambda_1^{kk_1}\lambda_2^{d_1k_1k_2},$$

implying that

$$\lambda_1^{k(k_2-k_1)} = \lambda_2^{d_1k_1k_2}.$$

Moreover, since $|\lambda_1| > 1$, $|\lambda_2| > 1$, the number $k_2 - k_1$ is positive. This proves the implication (7)⇒(30) in the case where A_1 and A_2 are not generalized Lattès maps.

Assume now that A_1, A_2 are arbitrary non-special rational functions. Then, by Theorem 2.2, there exist rational functions F_1, F_2, θ_1, θ_2 such that the diagrams

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F_1} & \mathbb{C} \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} \\ \mathbb{CP}^1 & \xrightarrow{A_1} & \mathbb{CP}^1 \end{array}, \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{F_2} & \mathbb{C} \\ \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2} \\ \mathbb{CP}^1 & \xrightarrow{A_2} & \mathbb{CP}^1 \end{array}$$

commute, and F_1, F_2 are not generalized Lattès maps. Further, since all the points in the preimage $\theta_{A_i}^{-1}\{z_i\}$, $i = 1, 2$, are F_i-preperiodic, there exist a positive integer N and fixed

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points z_1', z_2' of $F_1^{\circ N}$, $F_2^{\circ N}$ such that the diagrams

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F_1^{\circ N}} & \mathbb{C} \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} \\ \mathbb{CP}^1 & \xrightarrow{A_1^{\circ N}} & \mathbb{CP}^1 \end{array}, \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{F_2^{\circ N}} & \mathbb{C} \\ \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2} \\ \mathbb{CP}^1 & \xrightarrow{A_2^{\circ N}} & \mathbb{CP}^1 \end{array}$$

commute, and the equalities

$$\theta_1(z_1') = z_1, \quad \theta_1(z_2') = z_2$$

hold. Moreover, if μ_i is the multiplier of $F_i^{\circ N}$ at z_i', $i = 1, 2$, then, by Lemma 3.3, the equalities

$$\mu_1^{\mathrm{ord}_{z_1'}\theta_1} = \lambda_1^N, \quad \mu_2^{\mathrm{ord}_{z_2'}\theta_2} = \lambda_2^N, \tag{37}$$

$$\mathcal{P}_{A_1^{\circ N}, z_1, \lambda_1^N}(z^{\mathrm{ord}_{z_1'}\theta_1}) = \theta_1 \circ \mathcal{P}_{F_1^{\circ N}, z_1', \mu_1}(z), \tag{38}$$

$$\mathcal{P}_{A_2^{\circ N}, z_2, \lambda_2^N}(z^{\mathrm{ord}_{z_2'}\theta_2}) = \theta_2 \circ \mathcal{P}_{F_2^{\circ N}, z_2', \mu_2}(z) \tag{39}$$

hold.

Setting

$$f_1 = \mathrm{ord}_{z_1'}\theta_1, \quad f_2 = \mathrm{ord}_{z_2'}\theta_2, \quad f = f_1 f_2,$$

and substituting $z^{d_1 f_2}$ and $z^{d_2 f_1}$ for z into equalities (38) and (39), we obtain that

$$\mathcal{P}_{A_1, z_1, \lambda_1}(z^{d_1 f}) = \mathcal{P}_{A_1^{\circ N}, z_1, \lambda_1^N}(z^{d_1 f}) = \theta_1 \circ \mathcal{P}_{F_1^{\circ N}, z_1', \mu_1}(z^{d_1 f_2}),$$
$$\mathcal{P}_{A_2, z_2, \lambda_2}(z^{d_2 f}) = \mathcal{P}_{A_2^{\circ N}, z_2, \lambda_2^N}(z^{d_2 f}) = \theta_2 \circ \mathcal{P}_{F_2^{\circ N}, z_2', \mu_2}(z^{d_2 f_1}).$$

Thus, equality (7) implies that the functions $\mathcal{P}_{F_1^{\circ N}, z_1', \mu_1}(z^{d_1 f_2})$ and $\mathcal{P}_{F_2^{\circ N}, z_2', \mu_2}(z^{d_2 f_1})$ satisfy the equality

$$\widetilde{f}(\mathcal{P}_{F_1^{\circ N}, z_1', \mu_1}(z^{d_1 f_2}), \mathcal{P}_{F_2^{\circ N}, z_2', \mu_2}(z^{d_2 f_1})) = 0,$$

where

$$\widetilde{f}(x, y) = f(\theta_1(x), \theta_2(y)).$$

Since $F_1^{\circ N}$, $F_2^{\circ N}$ are not generalized Lattès maps, by what is proved above, there exist positive integers p_1, p_2 such that $\mu_1^{p_1} = \mu_2^{p_2}$, implying by equalities (37) that

$$\lambda_1^{p_1 f_2 N} = \mu_1^{p_1 f_1 f_2} = \mu_2^{p_2 f_1 f_2} = \lambda_2^{p_2 f_1 N}.$$

Thus, equality (30) holds for the integers

$$r_1 = p_1 f_2 N, \quad r_2 = p_2 f_1 N. \qquad \square$$

Proof of Corollary 1.2. If $\mathcal{P}_{A_1, z_1, \lambda_1}$, $\mathcal{P}_{A_2, z_2, \lambda_2}$ are algebraically dependent, then it follows from the commutativity of the diagram (8) that

$$(\deg A_1)^{l_1} = (\deg A_2)^{l_2} = \deg B,$$

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implying that $n_1^{l_1} = n_2^{l_2}$. Furthermore, it follows from equalities (28) that

$$\lambda_1^{l_1 \mathrm{ord}_{z_0} X_2} = \lambda_2^{l_2 \mathrm{ord}_{z_0} X_1}. \qquad \square$$

The following result shows that if A_1 and A_2 are not generalized Lattès maps, then dependencies of the form (7) actually reduce to dependencies of the form (1).

THEOREM 3.6. *Let A_1, A_2 be rational functions of degree at least two that are not generalized Lattès maps, z_1, z_2 their repelling fixed points with multipliers λ_1, λ_2, and $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ Poincaré functions. Assume that $C : f(x, y) = 0$ is an irreducible algebraic curve, and d_1, d_2 are coprime positive integers such that the equality*

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z^{d_1}), \mathcal{P}_{A_2,z_2,\lambda_2}(z^{d_2})) = 0$$

holds. Then, $d_1 = d_2 = 1$ and C has genus zero. Furthermore, if $C : f(x, y) = 0$ is an irreducible curve of genus zero with a generically one-to-one parameterization by rational functions $z \to (X_1(z), X_2(z))$, then the equality

$$f(\mathcal{P}_{A_1,z_1,\lambda_1}(z), \mathcal{P}_{A_2,z_2,\lambda_2}(z)) = 0$$

holds for some Poincaré functions $\mathcal{P}_{A_1,z_1,\lambda_1}$, $\mathcal{P}_{A_2,z_2,\lambda_2}$ if and only if there exist positive integers l_1, l_2 and a rational function B with a repelling fixed point z_0 such that the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1^{\circ l_1},A_2^{\circ l_2})} & (\mathbb{CP}^1)^2 \end{array}$$

commutes, and the equalities

$$\begin{aligned} X_1(z_0) &= z_1, \quad X_2(z_0) = z_2, \\ X_1'(z_0) &\neq 0, \quad X_2'(z_0) \neq 0 \end{aligned} \tag{40}$$

hold.

Proof. The proof is obtained by a modification of the proof of Theorem 1.1, taking into account that if A_1, A_2 are not generalized Lattès maps, then it follows from the commutativity of the diagram (8) by Proposition 2.1 that there exist rational functions Y_1 and Y_2 such that the equalities

$$Y_1 \circ X_1 = B^{\circ d_1} \quad Y_2 \circ X_2 = B^{\circ d_2}$$

hold for some $d_1, d_2 \geq 0$. Therefore, for any repelling fixed point z_0 of B, the inequalities (40) hold by the chain rule. Thus, $d_1 = d_2 = 1$ by equalities (10). $\square$

Notice that unlike the case of Böttcher functions, algebraic dependencies of the form (1) of degree greater than one between Poincaré functions do exist. The simplest of them are graphs constructed as follows. Let us take any two rational functions U and V, and set

$$A_1 = U \circ V, \quad A_2 = V \circ U. \tag{41}$$

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Then the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{A_1} & \mathbb{CP}^1 \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbb{CP}^1 & \xrightarrow{A_2} & \mathbb{CP}^1 \end{array}$$

obviously commutes. Moreover, if z_0 is a repelling fixed point of A_1, then the point $z_1 = V(z_0)$ is a repelling fixed point of A_2 by Lemma 3.3. Finally, the first equality in (41) implies that $V'(z_1) \neq 0$. Therefore,

$$\mathcal{P}_{A_2,z_2,\lambda_2} = V \circ \mathcal{P}_{A_1,z_1,\lambda_1},$$

by Lemma 3.3.

Notice also that the equality $d_1 = d_2 = 1$ provided by Theorem 3.6 does not hold for arbitrary non-special A_1, A_2. For example, let A be any rational function of the form $A = zR^d(z)$, where $R \in \mathbb{C}(z)$ and $d > 1$. Then one can easily check that $A : \mathcal{O} \to \mathcal{O}$, where $\mathcal{O}$ is defined by the equalities

$$\nu(0) = d, \quad \nu(\infty) = d,$$

is a minimal holomorphic map between orbifolds. Thus, A is a generalized Lattès map. Furthermore, the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{zR(z^d)} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle z^d} & & \downarrow{\scriptstyle z^d} \\ \mathbb{CP}^1 & \xrightarrow{zR^d(z)} & \mathbb{CP}^1 \end{array}$$

obviously commutes. Choosing now R in such a way that zero is a repelling fixed point of $zR(z^d)$ and denoting by λ the multiplier of $zR^d(z)$ at zero, we obtain by Lemma 3.3 that

$$\mathcal{P}_{zR^d(z),0,\lambda^d}(z^d) = z^d \circ \mathcal{P}_{zR(z^d),0,\lambda}(z).$$

Thus, $\mathcal{P}_{zR^d(z),0,\lambda^d}(z^d)$ and $\mathcal{P}_{zR(z^d),0,\lambda}(z)$ are algebraically dependent.

4. *Algebraic dependencies between Böttcher functions*

4.1. *Polynomial semiconjugacies and invariant curves.*

If A_1, A_2 are non-special *polynomials* of degree at least two, then any irreducible (A_1, A_2)-invariant curve C that is not a vertical or horizontal line has genus zero and allows for a generically one-to-one parameterization by *polynomials* X_1, X_2 such that the diagram

$$\begin{array}{ccc} (\mathbb{CP}^1)^2 & \xrightarrow{(B,B)} & (\mathbb{CP}^1)^2 \\ {\scriptstyle (X_1,X_2)}\downarrow & & \downarrow{\scriptstyle (X_1,X_2)} \\ (\mathbb{CP}^1)^2 & \xrightarrow{(A_1,A_2)} & (\mathbb{CP}^1)^2 \end{array} \tag{42}$$

commutes for some *polynomial* B (see [11, Proposition 2.34] or [17, §4.3]).

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For fixed polynomials A, B of degree at least two, we denote by $\mathcal{E}(A, B)$ the set (possibly empty) consisting of polynomials X of degree at least two such that the diagram (20) commutes. The following result was proved in [**17**] as a corollary of the results in [**15**].

THEOREM 4.1. *Let A and B be fixed non-special polynomials of degree at least two such that the set $\mathcal{E}(A, B)$ is non-empty, and let X_0 be an element of $\mathcal{E}(A, B)$ of the minimum possible degree. Then a polynomial X belongs to $\mathcal{E}(A, B)$ if and only if $X = \widetilde{A} \circ X_0$ for some polynomial $\widetilde{A}$ commuting with A.*

Notice that applying Theorem 4.1 for $B = A$, one can reprove the classification of commuting polynomials and, more generally, of commutative semigroups of $\mathbb{C}[z]$ obtained in [**5**, **28**, **29**] (see [**24**, §7.1], for more detail). On the other hand, applying Theorem 4.1 to the system of equation (42) with $A_1 = A_2 = A$, we see that X_1, X_2 cannot provide a generically one-to-one parameterization of C, unless one of the polynomials X_1, X_2 has degree one. Moreover if, say, X_1 has degree one, then without loss of generality, we may assume that $X_1 = z$, implying that $B = A$ and X_2 commutes with A. Thus, we obtain the following result obtained by Medvedev and Scanlon [**11**].

THEOREM 4.2. *Let A be a non-special polynomial of degree at least two, and C an irreducible algebraic curve that is not a vertical or horizontal line. Then, C is (A, A)-invariant if and only if C has the form $x = P(y)$ or $y = P(x)$, where P is a polynomial commuting with A.*

Finally, yet another corollary of Theorem 4.1 is the following result, which complements the classification of (A_1, A_2)-invariant curves obtained in [**11**] (see [**17**, Theorem 1.4]).

THEOREM 4.3. *Let A_1, A_2 be non-special polynomials of degree at least two, and C a curve. Then C is an irreducible (A_1, A_2)-invariant curve if and only if C has the form $Y_1(x) - Y_2(y) = 0$, where Y_1, Y_2 are polynomials of coprime degrees satisfying the equations*

$$T \circ Y_1 = Y_1 \circ A_1, \quad T \circ Y_2 = Y_2 \circ A_2$$

for some polynomial T.

4.2. *Proof of Theorem 1.3.* As in the case of Poincaré functions, we do not assume that considered Böttcher functions are normalized. Thus, the notation $\mathcal{B}_P$ is used to denote *some* function satisfying the conditions (4) and (5).

To prove Theorem 1.3, we need the following two lemmas.

LEMMA 4.4. *Let A, B be polynomials of degree at least two, and X a non-constant polynomial such that the diagram*

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle X} & & \downarrow{\scriptstyle X} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array}$$

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commutes. Assume that $\mathcal{B}_B$ *is a Böttcher function. Then,*

$$X \circ \mathcal{B}_B(z) = \mathcal{B}_A(z^{\deg X})$$

for some Böttcher function $\mathcal{B}_A$*.*

Proof. The lemma follows from [14, Lemma 2.1]. □

LEMMA 4.5. *Let A be a polynomial of degree* $n \geq 2$*, and* $\mathcal{B}_A$ *a Böttcher function. Assume that* $C : f(x, y) = 0$ *is an irreducible algebraic curve, and* d_1, d_2 *are positive integers such that* $d_1 \leq d_2$ *and the equality*

$$f(\mathcal{B}_A(z^{d_1}), \mathcal{B}_A(z^{d_2})) = 0 \tag{43}$$

holds. Then, C is a graph

$$P(x) - y = 0, \tag{44}$$

where P is a polynomial commuting with A, and the equality

$$d_1 \deg P = d_2 \tag{45}$$

holds.

Proof. Substituting z^n for z in equation (43), we see that the curve C is (A, A)-invariant. Therefore, by Theorem 4.2, C is a graph of the form $x = P(y)$ or $y = P(x)$, where P is a polynomial commuting with A. Taking into account that $d_1 \leq d_2$, this implies that equalities (44) and (45) hold. □

COROLLARY 4.6. *Let* A_1, A_2 *be polynomials of degree at least two, and* $\mathcal{B}_{A_1}, \mathcal{B}_{A_2}$ *Böttcher functions. Assume that* $C : f(x, y) = 0$ *is an irreducible algebraic curve of genus zero, and* $d_1, d_2, \widetilde{d}_1, \widetilde{d}_2$ *are positive integers such that* $\mathrm{GCD}(d_1, d_2) = 1$ *and the equalities*

$$f(\mathcal{B}_{A_1}(z^{d_1}), \mathcal{B}_{A_2}(z^{d_2})) = 0, \tag{46}$$

$$f(\mathcal{B}_{A_1}(z^{\widetilde{d}_1}), \mathcal{B}_{A_2}(z^{\widetilde{d}_2})) = 0 \tag{47}$$

hold. Then there exists a positive integer k such that the equalities

$$\widetilde{d}_1 = kd_1, \quad \widetilde{d}_2 = kd_2 \tag{48}$$

hold.

Proof. It is clear that equalities (46) and (47) imply the equalities

$$f(\mathcal{B}_{A_1}(z^{d_1\widetilde{d}_1}), \mathcal{B}_{A_2}(z^{d_2\widetilde{d}_1})) = 0 \tag{49}$$

and

$$f(\mathcal{B}_{A_1}(z^{d_1\widetilde{d}_1}), \mathcal{B}_{A_2}(z^{d_1\widetilde{d}_2})) = 0,$$

and eliminating from these equalities the function $\mathcal{B}_{A_1}(z^{d_1\widetilde{d}_1})$, we conclude that the functions $\mathcal{B}_{A_2}(z^{d_2\widetilde{d}_1})$ and $\mathcal{B}_{A_2}(z^{d_1\widetilde{d}_2})$ are algebraically dependent. Therefore, by Lemma 4.5, one of these functions is a polynomial in the other.

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Assume, say, that

$$\mathcal{B}_{A_2}(z^{d_2\widetilde{d}_1}) = R \circ \mathcal{B}_{A_2}(z^{d_1\widetilde{d}_2}) \tag{50}$$

(the other case is considered similarly). Then substituting the right part of this equality for the left part in equality (49), we conclude that

$$f(\mathcal{B}_{A_1}(z^{d_1\widetilde{d}_1}), R \circ \mathcal{B}_{A_2}(z^{d_1\widetilde{d}_2})) = 0,$$

implying that

$$f(\mathcal{B}_{A_1}(z^{\widetilde{d}_1}), R \circ \mathcal{B}_{A_2}(z^{\widetilde{d}_2})) = 0. \tag{51}$$

Let us observe now that equalities (47) and (51) imply that the curve $f(x, y) = 0$ is invariant under the map

$$(z_1, z_2) \to (\widehat{A}_1(z_1), \widehat{A}_2(z_2)) = (z_1, R(z_2)).$$

Since the commutativity of diagram (42) implies that $\deg A_1 = \deg A_2$, this yields that $\deg R = 1$. It follows now from equality (50) that

$$d_2\widetilde{d}_1 = d_1\widetilde{d}_2,$$

implying equality (48). □

Proof of Theorem 1.3. To prove the 'if' part, it is enough to observe that if equalities (12) and (13) hold, then by Lemma 4.4,

$$\begin{aligned} 0 = f(X_1, X_2) = f(X_1 \circ \mathcal{B}_B(z), X_2 \circ \mathcal{B}_B(z)) &= f(\mathcal{B}_{A_1}^{\circ l_1}(z^{\deg X_1}), \mathcal{B}_{A_2}^{\circ l_2}(z^{\deg X_2})) \\ &= f(\mathcal{B}_{A_1}(z^{\deg X_1}), \mathcal{B}_{A_2}(z^{\deg X_2})). \end{aligned}$$

In the other direction, if equality (11) holds, then setting $n_1 = \deg A_1$, $n_2 = \deg A_2$, and substituting z^{n_2} for z into equality (11), we obtain the equality

$$f(\mathcal{B}_{A_1}(z^{d_1 n_2}), A_2 \circ \mathcal{B}_{A_2}(z^{d_2})) = 0. \tag{52}$$

Eliminating now $\mathcal{B}_{A_2}(z^{d_2})$ from equalities (11) and (52), we conclude that the functions $\mathcal{B}_{A_1}(z^{d_1})$ and $\mathcal{B}_{A_1}(z^{d_1 n_2})$ are algebraically dependent. Since the corresponding algebraic curve $\widetilde{f}(x, y) = 0$ such that

$$\widetilde{f}(\mathcal{B}_{A_1}(z^{d_1}), \mathcal{B}_{A_1}(z^{d_1 n_2})) = 0$$

is (A_1, A_1)-invariant, it follows from Theorem 4.2 that

$$\mathcal{B}_{A_1}(z^{d_1 n_2}) = P \circ \mathcal{B}_{A_1}(z^{d_1}) \tag{53}$$

for some polynomial P commuting with A_1. Clearly, equality (53) implies that $\deg P = n_2$. On the other hand, by the Ritt theorem, P and A_1 have a common iterate. Therefore, there exist positive integers l_1, l_2 such that $n_1^{l_1} = n_2^{l_2}$.

Setting now

$$n = n_1^{l_1} = n_2^{l_2}$$

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and substituting z^n for z into equality (11), we obtain that $f(x, y) = 0$ is $(A_1^{\circ l_1}, A_2^{\circ l_2})$-invariant, implying that condition (12) holds. Moreover, by Theorem 4.3, $f(x, y) = 0$ has the form

$$Y_1(x) - Y_2(y) = 0, \tag{54}$$

where Y_1, Y_2 are polynomials of coprime degrees. Since a generically one-to-one parameterization $z \to (X_1(z), X_2(z))$ of curve (54) satisfies the conditions

$$\deg X_1 = \deg Y_2, \quad \deg X_2 = \deg Y_1,$$

we conclude that the degrees

$$\deg X_1 = d_1', \quad \deg X_2 = d_2'$$

of the functions X_1 and X_2 in diagram (12) satisfy $\mathrm{GCD}(d_1', d_2') = 1$. Using now the 'if' part of the theorem, we see that the equalities (11) and

$$f(\mathcal{B}_{A_1}(z^{d_1'}), \mathcal{B}_{A_2}(z^{d_2'})) = 0$$

hold simultaneously, implying by Corollary 4.6 that equalities $d_1' = d_1, d_2' = d_2$, and equality (13) hold. □

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